process, assuming no element used to eliminate those below it is zero, leads in the end to all elements below the leading diagonal of the first n columns of the modified augmented array becoming zero, so the final array becomes

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ 0 & a_{22}^{(1)} & \cdots & a_{2n}^{(1)} & b_2^{(1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{nn}^{(n-1)} & b_n^{(n-1)} \end{bmatrix}. \tag{32}$$

The solution is then found by the process called **back-substitution**, which starts with the last row in (32) that is equivalent to the equation $a_{nn}^{(n-1)}x_n = b_n^{(n-1)}$, from which it follows that

$$x_n = b_n^{(n-1)}/a_{nn}^{(n-1)}. \tag{33}$$

The second row from the bottom in (32) is equivalent to the equation

$$a_{n-1,n-1}^{(n-2)}x_{n-1} + a_{n-1,n}^{(n-2)}x_n = b_{n-1}^{(n-2)}, \tag{34}$$

from which x_{n-1} can be found after substituting the value of x_n found in (33). Continuing in this manner, all elements $x_1, x_2, \ldots, x_n$ of the required solution vector $\mathbf{x}$ can be found in the reverse order $x_n, x_{n-1}, \ldots, x_1$.

pivotal elements

The elements $a_{11}, a_{22}^{(1)}, \ldots, a_{nn}^{(n-1)}$ used to reduce the coefficient matrix $\mathbf{A}$ to the upper triangular form shown in the first n columns of (32) are called the **pivotal elements** in the Gaussian elimination process, and the row containing a pivotal element is called the **pivotal row**. This completes the basic Gaussian elimination process.

Clearly, if at the rth stage in the process a row of zeros is obtained in the modified coefficient matrix $\mathbf{A}$, but the modified rth element in the nonhomogeneous vector $\mathbf{b}$ is nonzero, the system of equations is incompatible and *no* solution exists. If, however, at the rth stage in the elimination process a row of zeros is obtained in the modified coefficient matrix $\mathbf{A}$, and the modified rth element in the nonhomogeneous vector $\mathbf{b}$ is also zero, then the rth equation is linearly dependent on the first $r - 1$ equations, so the solution cannot be unique.

A difficulty arises if at any stage of the process the pivotal element in the mth position on the leading diagonal of the modified matrix $\mathbf{A}$ becomes zero, as would happen at the start if $a_{11} = 0$. Should this occur, the difficulty is overcome by interchanging the order of the rows to bring a nonzero element into the pivotal position. Errors can be introduced during the elimination process if a very small pivotal element is used to reduce to zero entries in the column below it that are significantly larger, so this must be avoided. As the order of equations can be changed without altering the solution, these disadvantages can both be avoided as follows. At the mth stage, from among rows m to n, a row is selected that contains one of the elements of largest magnitude in its mth column. This row is then moved upward to form the new mth row, after which the elimination process continues as before. This process is called **Gaussian elimination with partial pivoting**, and it is a standard feature of software codes.

Gaussian elimination with partial pivoting

It is easy to see this same method can be used when the number of equations is not equal to the number of unknowns. The form of the modified augmented matrix will then, as just described, indicate whether the system has no solution, a unique

solution that can be found, or a nonunique solution depending on some arbitrary parameters because of linear independence of rows.

Gaussian elimination and det A

Although det $\mathbf{A}$ is not required when using the Gaussian elimination process, because the process reduces the original coefficient matrix $\mathbf{A}$ in an efficient manner to the upper-triangular form shown in the first n columns of (32), it follows at once that

$$\det \mathbf{A} = a_{11}a_{22}^{(1)}a_{33}^{(2)}\dots a_{nn}^{(n-1)}, \tag{35}$$

and it is this method that is used by software programs when finding det $\mathbf{A}$, thereby avoiding the many time-consuming multiplications involved when computing cofactors.

EXAMPLE 19.7

Solve the following system of equations by Gaussian elimination:

$$\begin{aligned} 2x_1 - 2x_2 + 3x_3 + 4x_4 &= -18 \\ 4x_1 + x_2 - x_3 + 2x_4 &= -11 \\ x_1 - x_2 - x_3 + 5x_4 &= -26 \\ 2x_1 - 3x_2 + 2x_3 - x_4 &= -3. \end{aligned}$$

Use (35) to find the determinant of the coefficient matrix $\mathbf{A}$.

Solution The array to be considered is

$$\begin{bmatrix} 2 & -2 & 3 & 4 & -18 \\ 4 & 1 & -1 & 2 & -11 \\ 1 & -1 & -1 & 5 & -26 \\ 2 & -3 & 2 & -1 & -3 \end{bmatrix},$$

in which the first four columns represent the coefficient matrix $\mathbf{A}$ and the last column the nonhomogeneous vector $\mathbf{b}$. As no element in the first column is small, there is no need to interchange rows, so we will use the entry $a_{11} = 2$ as the initial pivotal element. Subtracting twice the first row from the second row, half the first row from the third row, and the first row from the last row shows that at the end of the first stage of the Gaussian elimination process the modified array becomes

$$\begin{bmatrix} 2 & -2 & 3 & 4 & -18 \\ 0 & 5 & -7 & -6 & 25 \\ 0 & 0 & -\frac{5}{2} & 3 & -17 \\ 0 & -1 & -1 & -5 & 15 \end{bmatrix}.$$

The next element in the pivotal position is 5, so as this element is not small, the order of the rows can be left unchanged and the element 5 used as the next pivotal element. Adding one-fifth of row 2 to row 4 gives

$$\begin{bmatrix} 2 & -2 & 3 & 4 & -18 \\ 0 & 5 & -7 & -6 & 25 \\ 0 & 0 & -\frac{5}{2} & 3 & -17 \\ 0 & 0 & -\frac{12}{5} & -\frac{31}{5} & 20 \end{bmatrix}.$$

In the last stage of the elimination process we use $-5/2$ as the pivotal element and subtract 24/25 times row 3 from row 4 to obtain

$$\begin{bmatrix} 2 & -2 & 3 & 4 & -18 \\ 0 & 5 & -7 & -6 & 25 \\ 0 & 0 & -\frac{5}{2} & 3 & -17 \\ 0 & 0 & 0 & -\frac{227}{25} & \frac{908}{25} \end{bmatrix}.$$

Back substitution now gives the solution, because if we reinsert the unknown quantities $x_1, x_2, \ldots, x_n$ it follows from the last row that

$$-\frac{227}{25}x_4 = \frac{908}{25}, \quad \text{so } x_4 = -4,$$

while the second row from the bottom becomes

$$-\frac{5}{2}x_3 + 3x_4 = -17, \quad \text{so using } x_4 = -4 \text{ we find that } x_3 = 2.$$

Continuing in this manner and using the remaining two rows leads first to the result $x_2 = 3$ and then to $x_1 = -1$, so the solution is seen to be

$$x_1 = -1, \quad x_2 = 3, \quad x_3 = 2, \quad x_4 = -4.$$

Notice that in this case no pivotal element was small enough to necessitate an interchange of rows, so the solution was obtained without the need for partial pivoting.

The value of det **A** follows immediately from (35) as the product of the diagonal entries in the upper-triangular array to which the matrix **A** has been reduced at the end of the Gaussian elimination process, so

$$\det \mathbf{A} = 2 \cdot 5 \cdot \left(-\frac{5}{2}\right) \cdot \left(-\frac{277}{25}\right) = 277.$$ ■

The LU Factorization Method

Suppose the $n \times n$ nonsingular matrix **A** in the system $\mathbf{Ax} = \mathbf{b}$ can be factored as the product $\mathbf{A} = \mathbf{LU}$, where **L** is an $n \times n$ lower-triangular matrix with 1's along its leading diagonal and **U** is an $n \times n$ upper-triangular matrix.

The method of solution of the system of equations $\mathbf{Ax} = \mathbf{b}$ reduces to finding the column vector **y** that is the solution of $\mathbf{Ly} = \mathbf{b}$, and then determining **x** from the system of equations $\mathbf{Ux} = \mathbf{y}$. The advantage of this approach is that once **L** and **U** have been found, the elements of the vector **y** can be obtained by *forward substitution*, after which the elements of the vector **x** then follow by *backward substitution*. As already remarked, this approach is very efficient when the system $\mathbf{Ax} = \mathbf{b}$ has to be solved repeatedly with the same coefficient matrix **A**, but different nonhomogeneous vectors **b**. This is because **L** and **U** remain unchanged, so the solution vector **x** can be found using only multiplications, the vector **b**, and the known factorization of **A**. We remark here that, without introducing row permutations, it may not be possible to factor a nonsingular matrix.

All the information necessary for the factorization of **A** into the product $\mathbf{A} = \mathbf{LU}$ is already contained in the Gaussian elimination method, so the most straightforward form of **LU** factorization in which partial pivoting is not necessary will be illustrated by means of an example. We will factor the matrix **A** in Example 19.7, and then use the result to solve the system of equations in that example.

When the first stage of the Gaussian elimination process was applied to matrix **A** in the example, 2 times row 1 was *subtracted* from row 2, $\frac{1}{2}$ row 1 was *subtracted* from row 3, and 1 times row 1 was *subtracted* from row 4, causing matrix

$$\mathbf{A} = \begin{bmatrix} 2 & -2 & 3 & 4 \\ 4 & 1 & -1 & 2 \\ 1 & -1 & -1 & 5 \\ 2 & -3 & 2 & -1 \end{bmatrix} \quad \text{to become the matrix } \mathbf{A}_1 = \begin{bmatrix} 2 & -2 & 3 & 4 \\ 0 & 5 & -7 & -6 \\ 0 & 0 & -\frac{5}{2} & 3 \\ 0 & -1 & -1 & -5 \end{bmatrix}.$$

If we represent the elementary row operations involved in terms of premultiplication of **A** by a matrix $\mathbf{M}_1$, this can be written $\mathbf{M}_1\mathbf{A} = \mathbf{A}_1$, where

$$\mathbf{M}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -\frac{1}{2} & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}.$$

When the second stage of the Gaussian elimination process was applied to the matrix $\mathbf{A}_1$, $-\frac{1}{5}$ times row 2 was *subtracted* from row 4, causing $\mathbf{A}_2$ to become the matrix

$$\mathbf{A}_2 = \begin{bmatrix} 2 & -2 & 3 & 4 \\ 0 & 5 & -7 & -6 \\ 0 & 0 & -\frac{5}{2} & 3 \\ 0 & 0 & -\frac{12}{5} & -\frac{31}{5} \end{bmatrix},$$

so in terms of matrix multiplication this becomes $\mathbf{M}_2\mathbf{A}_1 = \mathbf{A}_2$, or $\mathbf{M}_2\mathbf{M}_1\mathbf{A} = \mathbf{A}_2$, where

$$\mathbf{M}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \frac{1}{5} & 0 & 1 \end{bmatrix}.$$

Finally, when the last stage of the Gaussian elimination process was applied to matrix $\mathbf{A}_2$, $24/25$ times row 3 was *subtracted* from row 4 to give the upper-triangular matrix

$$\mathbf{A}_3 = \begin{bmatrix} 2 & -2 & 3 & 4 \\ 0 & 5 & -7 & -6 \\ 0 & 0 & -\frac{5}{2} & 3 \\ 0 & 0 & 0 & -\frac{227}{25} \end{bmatrix},$$

so in terms of matrix multiplication $\mathbf{M}_3\mathbf{A}_2 = \mathbf{A}_3$, or $\mathbf{M}_3\mathbf{M}_2\mathbf{M}_1\mathbf{A} = \mathbf{A}_3$, where

$$\mathbf{M}_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{24}{25} & 1 \end{bmatrix}.$$

However, $\mathbf{A}_3 = \mathbf{U}$ is an upper-triangular matrix, and we have shown that

$$\mathbf{M}_3\mathbf{M}_2\mathbf{M}_1\mathbf{A} = \mathbf{U}, \quad \text{with } \mathbf{U} = \begin{bmatrix} 2 & -2 & 3 & 4 \\ 0 & 5 & -7 & -6 \\ 0 & 0 & -\frac{5}{2} & 3 \\ 0 & 0 & 0 & -\frac{227}{25} \end{bmatrix},$$

and so

$$\mathbf{A} = \mathbf{M}_1^{-1}\mathbf{M}_2^{-1}\mathbf{M}_3^{-1}\mathbf{U}.$$

We will have succeeded in factoring $\mathbf{A}$ if we can show that $\mathbf{M}_1^{-1}\mathbf{M}_2^{-1}\mathbf{M}_3^{-1}$ is a lower-triangular matrix of the required type.

To accomplish this last step notice that the special structure of the matrices $\mathbf{M}_i$, for $i = 1, 2, 3$ is such that from the definition of the inverse matrix in terms of its cofactors, the inverse matrix $\mathbf{M}_i^{-1}$ can be obtained directly from $\mathbf{M}_i$ by reversing the signs of the elements in its ith column that lie below the element 1, so without further computation we can write

$$\mathbf{M}_1^{-1}\mathbf{M}_2^{-1}\mathbf{M}_3^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -\frac{1}{5} & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{24}{25} & 1 \end{bmatrix}.$$

The structure of these matrices allows their product to be written down on sight, because the ith column of the product matrix is simply the ith column of the matrix $\mathbf{M}_i$, so that

$$\mathbf{M}_1^{-1}\mathbf{M}_2^{-1}\mathbf{M}_3^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & 1 & 0 \\ 1 & -\frac{1}{5} & \frac{24}{25} & 1 \end{bmatrix}.$$

This is a lower-triangular matrix of the required form, so

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & 1 & 0 \\ 1 & -\frac{1}{5} & \frac{24}{25} & 1 \end{bmatrix},$$

and the factored form of $\mathbf{A}$ is

$$\mathbf{A} = \mathbf{LU} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & 1 & 0 \\ 1 & -\frac{1}{5} & \frac{24}{25} & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 3 & 4 \\ 0 & 5 & -7 & -6 \\ 0 & 0 & -\frac{5}{2} & 3 \\ 0 & 0 & 0 & -\frac{227}{25} \end{bmatrix}.$$

To use $\mathbf{L}$ and $\mathbf{U}$ to solve the system of equations in Example 19.7, we must first solve the system $\mathbf{Ly} = \mathbf{b}$, where $\mathbf{b} = [-18, -11, -26, -3]^{\mathrm{T}}$. This is the system

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & 1 & 0 \\ 1 & -\frac{1}{5} & \frac{24}{25} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} -18 \\ -11 \\ -26 \\ -3 \end{bmatrix},$$

from which we see that $y_1 = -18$, and *forward substitution* then shows $y_2 = 25$, $y_3 = -17$, and $y_4 = 908/25$.

The elements x_1, x_2, x_3, and x_4 of the required solution vector $\mathbf{x}$ now follow by solving $\mathbf{Ux} = \mathbf{y}$, that is, the system

$$\begin{bmatrix} 2 & -2 & 3 & 4 \\ 0 & 5 & -7 & -6 \\ 0 & 0 & -\frac{5}{2} & 3 \\ 0 & 0 & 0 & -\frac{227}{25} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -18 \\ 25 \\ -17 \\ \frac{908}{25} \end{bmatrix}.$$

This shows $x_4 = -4$, so using *back substitution* we find that $x_3 = 2$, $x_2 = -3$, and $x_1 = -1$, so the system is solved.

This method has been described in its simplest form where straightforward Gaussian elimination is used without partial pivoting. The modification that is necessary to allow for row interchanges simply involves premultiplication at the appropriate stage by a permutation matrix. It will be recalled that a **permutation matrix** $\mathbf{P}$ is a matrix obtained from a unit matrix by interchanging its rows. If, for example, rows i and j of a unit matrix are interchanged to give the permutation matrix $\mathbf{P}$, then $\mathbf{PA}$ is the matrix obtained from $\mathbf{A}$ by interchanging its ith and jth rows. Use is then made of the result $\mathbf{PA} = \mathbf{LU}$.

An analysis of the steps involved in the foregoing approach leads to the following algorithm for the **LU** factorization of a nonsingular matrix $\mathbf{A}$ when no row interchanges are involved.

The LU factorization algorithm

the steps in LU factorization

The factorization of an $n \times n$ nonsingular matrix $\mathbf{A}$ into the product $\mathbf{A} = \mathbf{LU}$, where $\mathbf{L}$ is a lower-triangular matrix with 1's on its leading diagonal and $\mathbf{U}$ is an upper-triangular matrix, can be accomplished as follows.

1. The matrix $\mathbf{U}$ is obtained by applying the Gaussian elimination process to the rows of $\mathbf{A}$ to reduce it to an upper-triangular matrix.
2. At the ith stage of the Gaussian elimination process in Step 1, and in the ith column, let m_{ij} be the multiple of the ith element that must be *subtracted* from the jth element to reduce the jth element to zero. Then the matrix $\mathbf{L}$ is given by

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ m_{21} & 1 & 0 & \cdots & 0 & 0 \\ m_{31} & m_{32} & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & 1 & 0 \\ m_{n1} & m_{n2} & m_{n3} & \cdots & m_{nn-1} & 1 \end{bmatrix}.$$

EXAMPLE 19.8

Apply the **LU** factorization algorithm to determine the matrix $\mathbf{L}$ in Example 19.7.

Solution An examination of the Gaussian elimination process described in the example used to derive the algorithm shows that in the first step $m_{21} = 2$, $m_{31} = \frac{1}{2}$, and $m_{41} = 1$, and in the second step $m_{32} = 0$ and $m_{42} = -\frac{1}{5}$, while in the last step

$m_{43} = 24/25$, so from the algorithm

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & 1 & 0 \\ 1 & -\frac{1}{5} & \frac{24}{25} & 1 \end{bmatrix}.$$ ■

The Jacobi Iterative Process

To derive the Jacobi iterative process, the individual equations in (29) are rearranged so the first expresses x_1 in terms of the remaining unknowns and b_1, the second expresses x_2 in terms of the remaining unknowns and b_2, and so on until the last is rearranged to express x_n in terms of the other unknowns and b_n, leading to the result

$$\begin{aligned} x_1 &= (b_1 - a_{12}x_2 - a_{13}x_3 - \cdots - a_{1n}x_n)/a_{11} \\ x_2 &= (b_2 - a_{21}x_1 - a_{23}x_3 - \cdots - a_{2n}x_n)/a_{22} \\ &\cdots\cdots\cdots\cdots \\ x_n &= (b_n - a_{n1}x_1 - a_{n2}x_2 - \cdots - a_{n\,n-1}x_{n-1})/a_{nn}. \end{aligned} \tag{36}$$

Jacobi iterative method

The **Jacobi iterative process** follows from this by defining the rth approximation to the solution denoted by $x_1^{(r)}, x_2^{(r)}, \ldots, x_n^{(r)}$, in terms of the $(r-1)$th approximation denoted by $x_1^{(r-1)}, x_2^{(r-1)}, \ldots, x_1^{(r-1)}$, by means of the equations

$$\begin{aligned} x_1^{(r)} &= \left(b_1 - a_{12}x_2^{(r-1)} - a_{13}x_3^{(r-1)} - \cdots - a_{1n}x_n^{(r-1)}\right)/a_{11} \\ x_2^{(r)} &= \left(b_2 - a_{21}x_1^{(r-1)} - a_{23}x_3^{(r-1)} - \cdots - a_{2n}x_n^{(r-1)}\right)/a_{22} \\ x_3^{(r)} &= \left(b_3 - a_{31}x_1^{(r-1)} - a_{32}x_2^{(r-1)} - \cdots - a_{3n}x_n^{(r-1)}\right)/a_{33} \\ &\cdots\cdots\cdots\cdots \\ x_n^{(r)} &= \left(b_n - a_{n1}x_1^{(r-1)} - a_{n2}x_2^{(r-1)} - \cdots - a_{n\,n-1}x_{n-1}^{(r-1)}\right)/a_{nn}. \end{aligned} \tag{37}$$

The iteration is started with any initial choice for $x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)}$, typically $x_1^{(0)} = 1, x_2^{(0)} = 1, \ldots, x_n^{(0)} = 1$. The iterative process is continued until for some r the magnitude of the difference between corresponding elements of the $(r-1)$th and the rth iterates given by $|x_i^{(r)} - x_i^{(r-1)}|$ for $i = 1, 2, \ldots, n$ is less than some preassigned tolerance $\varepsilon > 0$, so that

$$\left|x_i^{(r)} - x_i^{(r-1)}\right| < \varepsilon, \quad \text{for } i = 1, 2, \ldots, n. \tag{38}$$

This is the simplest of many possible conditions for the **convergence** of an iterative process. The values $x_1^{(r)}, x_2^{(r)}, \ldots, x_n^{(r)}$ obtained from the rth iteration at which conditions (38) are first satisfied are taken to be the required solution $x_1, x_2, \ldots, x_n$, to within the tolerance ε. It should be noticed that the Jacobi iteration process is a fixed point iteration process for a system of linear equations.

Although it will not be proved here, a *sufficient* condition for the convergence of the Jacobi iterative process for any initial choice of $x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)}$ is that the

diagonal dominance

system (29) is **diagonally dominant**. This means that in each row of the coefficient matrix **A**, the magnitude of the element lying on the leading diagonal exceeds the sum of the magnitudes of all the other elements in the row. Thus, matrix **A** will be diagonally dominant if

$$|a_{ii}| > |a_{i1}| + |a_{i2}| + \cdots + |a_{ii-1}| + |a_{ii+1}| + \cdots + |a_{in}|, \quad \text{for } i = 1, 2, \ldots, n \tag{39}$$

An examination of the equations in (37) shows the Jacobi method fails to make use of current (improved) approximations as they are generated. This can be seen in the second equation where the better estimate $x_1^{(r)}$ could be used in place of the estimate $x_1^{(r-1)}$, as it has already been found from the first equation. Proceeding in this manner, and in each equation always using the currently available estimates, leads to the **Gauss–Seidel iterative process** defined by

Gauss–Seidel iterative method

$$\begin{aligned}
x_1^{(r)} &= \left(b_1 - a_{12}x_2^{(r-1)} - a_{13}x_3^{(r-1)} - \cdots - a_{1n}x_n^{(r-1)}\right)/a_{11} \\
x_2^{(r)} &= \left(b_2 - a_{21}x_1^{(r)} - a_{23}x_3^{(r-1)} - \cdots - a_{2n}x_n^{(r-1)}\right)/a_{22} \\
x_3^{(r)} &= \left(b_3 - a_{31}x_1^{(r)} - a_{32}x_2^{(r)} - \cdots - a_{3n}x_n^{(r-1)}\right)/a_{33} \\
&\cdots\cdots\cdots\cdots \\
x_n^{(r)} &= \left(b_n - a_{n1}x_1^{(r)} - a_{n2}x_2^{(r)} - \cdots - a_{nn-1}x_{n-1}^{(r)}\right)/a_{nn}.
\end{aligned} \tag{40}$$

A sufficient condition for the convergence of the Gauss–Seidel process is the same as that for the Jacobi process, namely that **A** is diagonally dominant.

spectral radius

Other conditions for the convergence of iterative processes can be derived in terms of the magnitude of the largest eigenvalue of **A**, called its **spectral radius**, but as this eigenvalue is difficult to compute when the number of equations n is large, such results are mainly of theoretical importance.

When an iterative process **diverges**, successive iterates usually alternate in sign and their magnitude grows without bound. In software programs a check is made on the behavior of successive iterates, and if divergence is detected the computer produces a message to this effect and terminates the computation.

EXAMPLE 19.9

Use the Gauss–Seidel iterative process to find the solution of the following system of equations

$$\begin{aligned}
1.2x_1 + 4.4x_2 - 1.9x_3 &= -4.2 \\
5.1x_1 - 1.3x_2 + 2.4x_3 &= 2.7 \\
-2.6x_1 + 1.7x_2 - 6.3x_3 &= 9.6.
\end{aligned}$$

Solution Applying the test for diagonal dominance in (39) shows that only the third equation satisfies the condition, because

$$|-6.3| > |-2.6| + |1.7| \quad \text{but} \quad |1.2| < |4.4| + |-1.9| \quad \text{and} \quad |-1.3| < |5.1| + |2.4|.$$

However, if the first two equations are interchanged the system becomes diagonally dominant, so when setting up the Gauss–Seidel iterative process in this case the

equations must be used in the order

$$\begin{aligned} 5.1x_1 - 1.3x_2 + 2.4x_3 &= 2.7 \\ 1.2x_1 + 4.4x_2 - 1.9x_3 &= -4.2 \\ -2.6x_1 + 1.7x_2 - 6.3x_3 &= 9.6. \end{aligned}$$

From (40) the Gauss–Seidel iterative process for this system of equations becomes

$$\begin{aligned} x_1^{(r)} &= \frac{1}{5.1}\left(1.3x_2^{(r-1)} - 2.4x_3^{(r-1)} + 2.7\right) \\ x_2^{(r)} &= \frac{1}{4.4}\left(-1.2x_1^{(r)} + 1.9x_3^{(r-1)} - 4.2\right) \\ x_3^{(r)} &= \frac{1}{6.3}\left(-2.6x_1^{(r)} + 1.7x_2^{(r)} - 9.6\right). \end{aligned}$$

The result of starting the iterations with $x_1^{(0)} = x_2^{(0)} = x_3^{(0)} = 1$ is shown in the following tables, and the values obtained in the 10th iteration should be compared with the solution $x_1 = 1.162946$, $x_2 = -2.418817$, and $x_3 = -2.656452$ obtained by Gaussian elimination.

Iteration Number					
	0	1	2	3	4	5
x_1	1	0.313726	1.229617	1.219913	1.175631	1.162857
x_2	1	−0.608289	−2.074693	−2.406137	−2.430951	−2.422740
x_3	1	−1.817425	−2.591108	−2.676541	−2.664962	−2.657887
	6	7	8	9	10	
x_1	1.162621	1.162815	1.162924	1.162946	1.162947	
x_2	−2.419348	−2.418785	−2.418784	−2.418809	−2.418816	
x_3	−2.656461	−2.656389	−2.656434	−2.656450	−2.656452	

These results demonstrate the *convergence* of the iterations obtained from a diagonally dominant scheme to the solution obtained by the direct method.

how nondiagonal dominance can lead to divergence

If, instead, an iterative scheme had been derived from the original system of equations without first rearranging them to make the system diagonally dominant, we would have obtained

$$\begin{aligned} x_1^{(r)} &= \frac{1}{1.2}\left(-4.4x_2^{(r-1)} + 1.9x_3^{(r-1)} - 4.2\right) \\ x_2^{(r)} &= \frac{1}{1.3}\left(5.1x_1^{(r)} + 2.4x_3^{(r-1)} - 2.7\right) \\ x_3^{(r)} &= \frac{1}{6.3}\left(-2.6x_1^{(r)} + 1.7x_2^{(r)} - 9.6\right). \end{aligned}$$

Using this scheme, and starting the iterations as before with $x_1^{(0)} = x_2^{(0)} = x_3^{(0)} = 1$, gives the results

$$\begin{aligned} &x_1^{(1)} = -5.58333, \quad x_2^{(1)} = -22.13462, \quad x_3^{(1)} = -5.19241 \\ &x_1^{(2)} = 69.43894, \quad x_2^{(2)} = 260.75140, \quad x_3^{(2)} = 40.18034 \end{aligned}$$

that demonstrate very clearly the *divergence* of the nondiagonally dominant scheme. ■

Something must be said about how these two iterative methods are used. The Gauss–Seidel method is used in computer codes mainly as a preconditioner for more advanced schemes, where its use of the current approximation at each stage requires only half as much storage as the Jacobi method. The Jacobi schemes are used extensively as building blocks in much more complicated and efficient iterative procedures, such as preconditioned conjugate gradient and multigrid methods.

For more information about numerical linear algebra, see references [2.15], [2.16], [2.17], [2.19], and [2.20].

Summary

Various examples were given, and it was seen that the **LU** factorization of an $n \times n$ matrix **A** is only possible if det $\mathbf{A} \neq 0$.

Two essentially different types of methods have been derived for the solution of systems of nonhomogeneous linear equations, one of a direct type and the other based on iteration. The two direct methods were Gaussian elimination and the **LU** factorization method that is derived from it. The necessity to interchange rows when a pivotal element was either zero or small was shown to lead to Gaussian elimination with partial pivoting. The **LU** factorization method was shown to make use of the information produced by the Gaussian elimination process at each step in a different manner, and it may also involve partial pivoting.

The other method, involving iteration, started from an arbitrary initial approximation and converged to the required solution to within a prescribed tolerance, provided the system of equations was diagonally dominant.

EXERCISES 19.5

In Exercises 1 through 4, (a) solve the system of equations using Gaussian elimination, and (b) compare the results obtained in (a) with those found by solving the system using Gauss–Seidel iteration starting from the initial iterates $x_1^{(0)} = x_2^{(0)} = x_3^{(0)} = 1$ and performing 10 iterations.

1. $4.7x_1 + 1.3x_2 - 1.6x_3 = 1.3$
$x_1 - 4.1x_2 + 1.1x_3 = 4.6$
$2.1x_1 + 1.4x_2 + 6.2x_3 = 5.2.$

2. $1.7x_1 - 4.6x_2 - 1.2x_3 = 3.4$
$-3.1x_1 + 2.3x_2 + 7.2x_3 = 2.7$
$3.2x_1 + 1.2x_2 + 1.4x_3 = -4.2.$

3. $2.1x_1 + 6.5x_2 - 3.1x_3 = -6.4$
$-5.2x_1 + 2.1x_2 - 1.5x_3 = 3.7$
$1.8x_1 - 2.9x_2 + 6.2x_3 = -4.2.$

4. $6.2x_1 - 2.2x_2 + 3.1x_3 = -2.6$
$-1.6x_1 + 1.9x_2 + 8.4x_3 = -2.6$
$2.3x_1 - 8.4x_2 + 3.2x_3 = 6.5.$

5. The $n \times n$ real symmetric matrix $\mathbf{H}_n$ with the element $h_{ij} = 1/(i + j - 1)$ in its ith row and jth column is called the **Hilbert matrix**, and its determinant rapidly becomes vanishingly small as n increases. Matrices of this type are said to be **ill-conditioned**, and when ill-conditioned matrices occur as coefficient matrices in systems of linear equations, large errors arise unless the calculations are performed using very high precision. The development of a vanishingly small determinant of a Hilbert matrix can be seen, for example, even when $n = 4$, because

$$\mathbf{H}_4 = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \end{bmatrix}, \text{ and } \det \mathbf{H}_4 = 1/6{,}048{,}000.$$

When the fractions involved are not approximated, the exact solution of the system of equations

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

can be shown to be $x_1 = -64$, $x_2 = 900$, $x_3 = -2520$, and $x_4 = 1820$. Typically, ill-conditioned matrices arise in least squares approximations and orthogonalization.

Demonstrate the errors that arise when Gaussian elimination is used to solve this system of equations and the calculations are rounded to five decimal places. Use the Gaussian elimination to calculate det $\mathbf{H}_4$ working to five decimal places and compare the value obtained with the true result.

6. Use Jacobi and Gauss–Seidel iteration to solve the system

$$\begin{aligned} -4.2x_1 + 1.1x_2 - 2.1x_3 &= 1.4 \\ 3.6x_1 + 9.2x_2 - 3.1x_3 &= -3.2 \\ 1.4x_1 + 2.9x_2 - 6.4x_3 &= -1.2, \end{aligned}$$

starting from the initial iterates $x_1^{(0)} = x_2^{(0)} = x_3^{(0)} = 0$ and performing six iterations. Compare the results with the exact solution $x_1 = -0.39101$, $x_2 = -0.18938$, $x_3 = 0.01615$. Derive an iterative scheme when the equations are arranged in a *nondiagonally dominant* form, and using the initial iterates $x_1^{(0)} = x_2^{(0)} = x_3^{(0)} = 0$ perform three iterations to demonstrate the divergence of the scheme.

In Exercises 7 through 12 use **LU** factorization to solve the system of equations $\mathbf{Ax} = \mathbf{b}$ for the given matrices $\mathbf{A}$ and $\mathbf{b}$.

7. $\mathbf{A} = \begin{bmatrix} -4 & 1 & -1 \\ 12 & -1 & 5 \\ -12 & 5 & -4 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix}$.

8. $\mathbf{A} = \begin{bmatrix} -1 & 2 & 3 \\ -5 & 7 & 16 \\ 2 & -10 & -2 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -5 \\ 2 \\ 6 \end{bmatrix}$.

9. $\mathbf{A} = \begin{bmatrix} 4 & -1 & -1 \\ -16 & 6 & 1 \\ -4 & 7 & -9 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 0 \\ 6 \\ -7 \end{bmatrix}$.

10. $\mathbf{A} = \begin{bmatrix} -5 & -2 & 0 \\ -15 & -9 & 2 \\ 0 & -6 & 8 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$.

11. $\mathbf{A} = \begin{bmatrix} 2 & 1 & 0 & 2 \\ -1 & 0 & 1 & 0 \\ 4 & \frac{3}{2} & 2 & 3 \\ -2 & 0 & 8 & 1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 4 \end{bmatrix}$.

12. $\mathbf{A} = \begin{bmatrix} 3 & 0 & 1 & -1 \\ 6 & -1 & 3 & -3 \\ -3 & 1 & 0 & 1 \\ -3 & 0 & -5 & 4 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -2 \\ 3 \\ -1 \\ 5 \end{bmatrix}$.

19.6 Eigenvalues and Eigenvectors

In Chapter 4 an **eigenvalue** associated with an $n \times n$ matrix $\mathbf{A}$ was defined as a number λ satisfying the matrix equation

$$\mathbf{Ax} = \lambda\mathbf{x}, \tag{41}$$

and the corresponding $n \times 1$ vector $\mathbf{x}$ was defined as the associated **eigenvector**. It follows directly from (41) that an eigenvector $\mathbf{x}$ of $\mathbf{A}$ corresponding to an eigenvalue λ can be multiplied (scaled) by a nonzero number k and still remain an eigenvector, because

$$\mathbf{A}(k\mathbf{x}) = \lambda(k\mathbf{x}) \quad \text{is equivalent to} \quad k\mathbf{Ax} = k\lambda\mathbf{x},$$

and cancellation of the scalar k reduces this last result to (41).

When eigenvalues and eigenvectors were determined in Chapter 4, result (41) was rewritten as the homogeneous system $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$, and the eigenvalues were found by requiring the determinant of the coefficient matrix $\det(\mathbf{A} - \lambda\mathbf{I})$ to vanish, leading to a polynomial in λ of degree n of the general form

$$P(\lambda) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_n,$$

called the **characteristic polynomial** associated with $\mathbf{A}$. Once the zeros of $P(\lambda)$ had been found, that is, the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of $\mathbf{A}$, the associated eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ were then obtained by solving the matrix equation

$$\mathbf{Ax}_i = \lambda_i\mathbf{x}_i \quad \text{for } i = 1, 2, \ldots, n. \tag{42}$$

This theoretical approach is only useful when $n \leq 3$, because then the zeros of $P(\lambda)$ can be determined analytically. In all other cases the task of finding the zeros is difficult, and unless they are known accurately, significant errors can be introduced when using them in (42) to compute the associated eigenvectors.

Computationally efficient numerical methods are available in computer algebra packages for the determination of eigenvalues and eigenvectors that do not involve first solving the characteristic equation for the eigenvalues. These are capable of finding real and complex eigenvalues, including repeated eigenvalues, and the corresponding eigenvectors. Because of this the only method that will be described here will be the **power method**, as it is easy to apply and its derivation is straightforward. However, this is not the method that is used in practice, except in certain special situations.

The derivation requires all of the eigenvalues of $\mathbf{A}$ to be ordered according to absolute magnitude so that

$$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_n|. \tag{43}$$

When this ordering is adopted, the eigenvalue λ_1 with the greatest magnitude is called the **dominant** eigenvalue of matrix $\mathbf{A}$, and the remaining eigenvalues $\lambda_2, \lambda_3, \ldots, \lambda_n$ are then called the **subdominant** eigenvalues of $\mathbf{A}$.

dominant and subdominant eigenvalues

It was seen in Chapter 4 that an arbitrary n element column vector $\mathbf{v}_0$ can always be expressed as the linear combination of eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$,

$$\mathbf{v}_0 = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n, \tag{44}$$

the power method for the dominant eigenvalue and its eigenvector

for some suitable choice of constants $c_1, c_2, \ldots, c_n$. The **power method** for the simultaneous determination of the eigenvalues and eigenvectors of $\mathbf{A}$ is an *iterative* method, and it involves setting $\mathbf{v}_r = \mathbf{A}^r\mathbf{v}_0$, multiplying (44) by $\mathbf{A}^r$, and making use of results (42) and (43). For $r = 0, 1, 2, \ldots,$ we have

$$\begin{aligned}\mathbf{v}_r &= \mathbf{A}^r(c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n)\\ &= c_1\lambda_1^r\mathbf{x}_1 + c_2\lambda_2^r\mathbf{x}_2 + \cdots + c_n\lambda_n^r\mathbf{x}_n\\ &= \lambda_1^r\{c_1\mathbf{x}_1 + c_2(\lambda_2/\lambda_1)^r\mathbf{x}_2 + \cdots + c_n(\lambda_n/\lambda_1)^r\mathbf{x}_n\}.\end{aligned} \tag{45}$$

The ordering of the eigenvalues in (43) causes the factors $(\lambda_2/\lambda_1)^r, (\lambda_3/\lambda_1)^r, \ldots,$ $(\lambda_n/\lambda_1)^r$ in (45) all to tend to zero as r increases, so assuming that $c_1 \neq 0$, for suitably large r equation (45) can be approximated by

$$\mathbf{x}_r \approx \lambda_1^r c_1\mathbf{x}_1. \tag{46}$$

The assumption that $c_1 \neq 0$ is not restrictive, because if this is true, roundoff can be expected to introduce a component in the direction of $\mathbf{x}_1$, so that although convergence will be delayed, it will still take place in practice.

Result (46) shows that when r is large, $\mathbf{v}_r$ can be taken to be proportional to the eigenvector $\mathbf{x}_1$ associated with the dominant eigenvalue λ_1. As $\mathbf{v}_r = \mathbf{A}^r\mathbf{v}_0 = \mathbf{A}(\mathbf{A}^{r-1}\mathbf{v}_0) = \mathbf{A}\mathbf{v}_{r-1}$, it follows that the ratio (quotient) of corresponding elements in $\mathbf{v}_r$ and $\mathbf{v}_{r-1}$ approximate the dominant eigenvalue λ_1.

When the power method is implemented, the elements in $\mathbf{v}_r$ can become very large or very small, so to keep the exponent range of the machine from being exceeded, the fact that an eigenvector can be scaled and still remain an eigenvector

is used to redefine the vector $\mathbf{v}_r$ as $\mathbf{v}_r = \mathbf{A}\widetilde{\mathbf{v}}_{r-1}$, where $\widetilde{\mathbf{v}}_{r-1}$ is a *normalized* vector $\mathbf{v}_{r-1}$. Many normalizations are possible, but the most convenient one involves obtaining $\widetilde{\mathbf{v}}_{r-1}$ from $\mathbf{v}_{r-1}$ by dividing each element of $\mathbf{v}_{r-1}$ by α_{r-1}, where α_{r-1} is its element of greatest magnitude. As a result of this normalization $\mathbf{v}_{r-1} = \alpha_{r-1}\widetilde{\mathbf{v}}_{r-1}$, and the element in $\widetilde{\mathbf{v}}_{r-1}$ with greatest magnitude becomes 1.

The iteration equation $\mathbf{v}_r = \mathbf{A}\mathbf{v}_{r-1}$ must now be replaced by $\mathbf{v}_r = \mathbf{A}\widetilde{\mathbf{v}}_{r-1}$ for $r = 1, 2, \ldots,$ or, equivalently, by

normalization of vectors

$$\mathbf{v}_{r+1} = \mathbf{A}\widetilde{\mathbf{v}}_r \quad \text{for } r = 0, 1, \ldots. \tag{47}$$

Substituting $\mathbf{v}_{r+1} = \alpha_{r+1}\widetilde{\mathbf{v}}_{r+1}$ in the preceding result gives $\mathbf{A}\widetilde{\mathbf{v}}_r = \alpha_{r+1}\widetilde{\mathbf{v}}_{r+1}$, so as r becomes large and $\widetilde{\mathbf{v}}_r \to \widetilde{\mathbf{v}}_{r+1}$, it follows that $\alpha_{r+1} \to \lambda_1$ and $\widetilde{\mathbf{v}}_{r+1} \to \widetilde{\mathbf{x}}_1$, the normalized eigenvector associated with λ_1. The iteration process in (47) can be started with any constant vector $\mathbf{v}_0 = [v_1, v_2, \ldots, v_n]^{\mathrm{T}}$ that is often taken to be $\mathbf{v}_0 = [1, 1, \ldots, 1]^{\mathrm{T}}$. The rate of convergence of the iterations is fastest when $|\lambda_1| \gg |\lambda_2|$, but the convergence becomes very slow when $|\lambda_1|$ and $|\lambda_2|$ are close together.

Various methods exist for the determination of the subdominant eigenvalues once the dominant eigenvalue is known, though these will not be discussed here.

EXAMPLE 19.10

Use the power method to find the dominant eigenvalue λ_1 and the normalized eigenvector $\mathbf{x}_1$ when

$$\mathbf{A} = \begin{bmatrix} 1 & 4 & 1 & 2 \\ 4 & 0 & 3 & 1 \\ 1 & 3 & 1 & 2 \\ 2 & 1 & 2 & 1 \end{bmatrix}.$$

Solution As the matrix $\mathbf{A}$ is symmetric, its eigenvalues will all be real, so it is appropriate to use the power method to determine its eigenvalues and eigenvectors. In order to determine the dominant eigenvalue and its associated eigenvector, the iterative process $\mathbf{v}_r = \mathbf{A}\widetilde{\mathbf{v}}_{r-1}$ will be started by setting $\mathbf{v}_0 = [1, 1, 1, 1]^{\mathrm{T}}$, and in the table that follows the ith element of $\mathbf{v}_r$ is denoted by $v_r^{(i)}$ while the corresponding normalized ith element of $\widetilde{\mathbf{v}}_r$ is denoted by $\widetilde{v}_r^{(i)}$.

Iterations Using $\mathbf{v}_{r+1} = \mathbf{A}\widetilde{\mathbf{v}}_r$

Iteration r	0	1	2	3	4	5	6	7	8	9	10
$v_r^{(1)}$	1	8	7.375	7.35593	7.34334	7.35018	7.34608	7.34881	7.34748	7.34770	7.34756
$v_r^{(2)}$	1	8	7.375	7.33899	7.33642	7.33732	7.33674	7.33797	7.33695	7.33696	7.33695
$v_r^{(3)}$	1	7	6.375	6.35593	6.34569	6.35112	6.34783	6.35008	6.34896	6.34913	6.34902
$v_r^{(4)}$	1	6	5.5	5.47458	5.47006	5.47224	5.47091	5.47229	5.47137	5.47143	5.47135
α_r	1	8	7.375	7.35593	7.34334	7.35018	7.34608	7.34881	7.34748	7.34770	7.34756
$\widetilde{v}_r^{(1)}$	1	1	1	1	1	1	1	1	1	1	1
$\widetilde{v}_r^{(2)}$	1	1	1	0.99770	0.99906	0.99825	0.99873	0.99852	0.99857	0.99854	0.99856
$\widetilde{v}_r^{(3)}$	1	0.87500	0.86441	0.86406	0.86414	0.86408	0.86411	0.86410	0.86410	0.86410	0.86410
$\widetilde{v}_r^{(4)}$	1	0.75000	0.74576	0.74424	0.74490	0.74450	0.74474	0.74465	0.74466	0.74465	0.74465

This shows that after 10 iterations the approximation to λ_1 provided by α_1 is $\lambda_1 \approx 7.34756$, and the associated normalized eigenvector $\widetilde{\mathbf{x}}_1$ is

$$\widetilde{\mathbf{x}}_1 \approx [1, 0.99856, 0.86410, 0.74465]^{\mathrm{T}}.$$

A calculation using a software package shows that when approximated to five decimal places $\lambda_1 = 7.34760$ and $\widetilde{\mathbf{v}}_1 = [1, 0.99855, 0.86410, 0.74465]^{\mathrm{T}}$. ■

Euclidean norm of a vector

A different normalization that is often used involves dividing a vector $\mathbf{u}$ by $\|\mathbf{u}\| = (u_1^2 + u_2^2 + \cdots + u_n^2)^{1/2}$, where $u_1, u_2, \ldots, u_n$ are the n elements of $\mathbf{u}$. $\|\mathbf{u}\|$ is called the **Euclidean norm**, and it is useful when working with eigenvalues and eigenvectors of symmetric matrices, because then the quotient of corresponding terms in successive iterations provides a higher order approximation to the eigenvalue.

the inverse power method and finding the eigenvalue closest to a given number

The power method can also be used to find the eigenvalue λ_n of an $n \times n$ matrix $\mathbf{A}$ with the *smallest* magnitude, together with its associated eigenvector. The idea is simple, and it starts from the fact that if $\mathbf{A}$ is a nonsingular $n \times n$ matrix with the real eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, then these are solutions of $\mathbf{Ax} = \lambda\mathbf{x}$. As $\mathbf{A}$ is nonsingular, it has an inverse $\mathbf{A}^{-1}$, and premultiplication of $\mathbf{Ax} = \lambda\mathbf{x}$ by $\mathbf{A}^{-1}$ gives $\mathbf{A}^{-1}\mathbf{Ax} = \lambda\mathbf{A}^{-1}\mathbf{x}$, or $\mathbf{A}^{-1}\mathbf{x} = (1/\lambda)\mathbf{x}$, showing that $1/\lambda_1, 1/\lambda_2, \ldots, 1/\lambda_n$ are the eigenvalues of $\mathbf{A}^{-1}$ and that the eigenvectors associated with λ_i and $1/\lambda_i$ are identical. Consequently, if the eigenvalues are ordered so that $|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_n|$, the eigenvalue of $\mathbf{A}$ with the *smallest* magnitude will be the *dominant* eigenvalue of $\mathbf{A}^{-1}$. Thus, an application of the power method to $\mathbf{A}^{-1}$ will generate its dominant eigenvalue $\mu_1 = 1/\lambda_n$, so that $\lambda_n = 1/\mu_1$.

When using this method the inverse matrix $\mathbf{A}^{-1}$ is *not* constructed, and instead the equation

$$\boxed{\mathbf{Av}_{r+1} = \mathbf{v}_r} \tag{48}$$

is iterated, having first used $\mathbf{LU}$ decomposition to solve for $\mathbf{v}_{r+1}$ in terms of $\mathbf{v}_r$. The decomposition only needs to be performed once because afterwards, at each stage of the iteration, the elements of $\mathbf{v}_{r+1}$ can be found by back-substitution using the elements of $\mathbf{v}_r$. This is just the situation where an $\mathbf{LU}$ decomposition is needed, because the right-hand sides are not available in advance, so it is necessary to solve a sequence of problems with the same matrix. Without the $\mathbf{LU}$ decomposition this process is not really practical.

As with the previous iteration procedure it is again necessary to normalize $\mathbf{v}_r$ by dividing each of its elements by its element of greatest magnitude α_r, or to use some other form of normalization, to keep calculations within the exponent range of the machine. This is because, unlike the previous case where the nonnormalized elements of $\mathbf{v}_r$ increased in magnitude as r increased, in this case they will decrease, causing accuracy to be lost if normalization is not performed. This method is called the **inverse power method** because it is equivalent to iterating the inverse matrix $\mathbf{A}^{-1}$. If we denote the normalized column vector $\mathbf{v}_r$ by $\widetilde{\mathbf{v}}_r$, the iteration scheme to be used analogous to (47) becomes

$$\boxed{\mathbf{Av}_{r+1} = \widetilde{\mathbf{v}}_r \quad \text{for } r = 0, 1, \ldots.} \tag{49}$$

EXAMPLE 19.11 Use the inverse power method to find the eigenvalue of $\mathbf{A}$ with the smallest magnitude, given that

$$\mathbf{A} = \begin{bmatrix} 4 & 2 & 4 \\ 3 & 9 & 2 \\ 5 & 6 & 9 \end{bmatrix}.$$

Solution The required eigenvalue will be obtained by iterating $\mathbf{A}\tilde{\mathbf{v}}_{r+1} = \mathbf{v}_r$ with the given matrix $\mathbf{A}$, so the system to be considered is

$$\begin{bmatrix} 4 & 2 & 4 \\ 3 & 9 & 2 \\ 5 & 6 & 9 \end{bmatrix} \begin{bmatrix} v_1^{(r+1)} \\ v_2^{(r+1)} \\ v_3^{(r+1)} \end{bmatrix} = \begin{bmatrix} \tilde{v}_1^{(r)} \\ \tilde{v}_2^{(r)} \\ \tilde{v}_3^{(r)} \end{bmatrix} \quad \text{with } r = 0, 1, \ldots \text{ and } \begin{bmatrix} v_1^{(0)} \\ v_2^{(0)} \\ v_3^{(0)} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Using **LU** decomposition the system becomes

$$\begin{aligned} 4v_1^{(r+1)} + 2v_2^{(r+1)} + 4v_3^{(r+1)} &= \tilde{v}_1^{(r)} \\ \frac{15}{2}v_2^{(r+1)} - v_3^{(r)} &= \tilde{v}_2^{(r)} \\ \frac{67}{15}v_3^{(r)} &= \tilde{v}_3^{(r)} \end{aligned}$$

and $\mathbf{v}_{r+1}$ now follows from $\tilde{\mathbf{v}}_r$ by back-substitution. As r increases, so the ratio of corresponding components of $\tilde{\mathbf{v}}_{r+1}$ and $\tilde{\mathbf{v}}_r$ will tend to the eigenvalue μ_1 of $\mathbf{A}^{-1}$ of greatest magnitude, so that the eigenvalue of $\mathbf{A}$ of *smallest* magnitude will be $\lambda_3 = 1/\mu_1$. The results of eight iterations are listed below, as in Example 19.10.

Iterations Using $\mathbf{A}\mathbf{v}_{r+1} = \tilde{\mathbf{v}}_r$

Iteration	0	1	2	3	4	5	6	7	8
$v_r^{(1)}$	1	0.32090	0.57914	0.61488	0.61215	0.60984	0.60898	0.60871	0.60862
$v_r^{(2)}$	1	0.02239	−0.12617	−0.16659	−0.17289	−0.17403	−0.17429	−0.17436	−0.17438
$v_r^{(3)}$	1	−0.08209	−0.26606	−0.28158	−0.27571	−0.27282	−0.27183	−0.27152	−0.27143
α_r	1	0.32090	0.57914	0.61488	0.61215	0.60984	0.60898	0.60871	0.60862
$\tilde{v}_r^{(1)}$	1	1	1	1	1	1	1	1	1
$\tilde{v}_r^{(2)}$	1	0.06977	−0.21786	−0.27093	−0.28243	−0.28637	−0.28620	−0.28644	−0.28652
$\tilde{v}_r^{(3)}$	1	−0.25582	−0.45941	−0.45794	−0.45040	−0.44736	−0.44637	−0.44606	−0.44598

This shows that the approximate value of the *largest* eigenvalue of $\mathbf{A}^{-1}$ given by α_8 is $\mu_1 \approx 0.60862$, so the approximate value of the *smallest* eigenvalue of $\mathbf{A}$ is $\lambda_3 = 1/\mu_1 = 1.64306$, and the corresponding approximation to the associated normalized eigenvector $\mathbf{x}_3$ provided by $\mathbf{v}_8$ is

$$\mathbf{x}_3 \approx [1, -0.28652, -0.44598]^{\mathrm{T}}.$$

The results accurate to five decimal places found by using a software package are $\lambda_3 = 1.64315$ and $\mathbf{x}_3 = [1, -0.28656, -0.44592]^{\mathrm{T}}$. ■

As an extension of the previous argument, let k be a specified constant, and consider the matrix $\mathbf{B} = \mathbf{A} - k\mathbf{I}$. Then, in terms of matrix $\mathbf{B}$, the eigenvalue equation

$\mathbf{A}\mathbf{x}_i = \lambda_i \mathbf{x}_i$ becomes

$$\mathbf{B}\mathbf{x}_i = (\lambda_i - k)\mathbf{x}_i, \tag{50}$$

showing the eigenvectors of $\mathbf{A}$ and $\mathbf{B}$ are identical, but the eigenvalues $\lambda_i - k$ of $\mathbf{B}$ are those of $\mathbf{A}$ reduced by k. This means that the eigenvalues of $(\mathbf{A} - k\mathbf{I})^{-1}$ for $k \neq \lambda_i$, with $i = 1, 2, \ldots, n$, are $1/(\lambda_1 - k), 1/(\lambda_2 - k), \ldots, 1/(\lambda_n - k)$. An application of the inverse power method to $(\mathbf{A} - k\mathbf{I})^{-1}$ then determines the eigenvalue of $\mathbf{A}$ closest to the specified constant k. This can be used as a basis for computing an eigenvector once an eigenvalue has been found. In terms of this approach, the initial application of the inverse power method is seen to involve the determination of the eigenvalue of $\mathbf{A}$ closest to 0.

For more information about the numerical computation of eigenvalues and eigenvectors see references [2.15], [2.16], [2.17], [2.19], and [2.20].

Summary

The power method for the calculation of the eigenvalue of greatest magnitude of a matrix together with its associated eigenvector was described. It was then shown how the inverse power method can be used to find the eigenvalue of smallest magnitude, and by making a small modification to the inverse power method, how the eigenvalue closest to a given number k can be found.

EXERCISES 19.6

In Exercises 1 through 4 use the power method to find the approximate value of the dominant eigenvalue and the associated normalized eigenvector of the given matrix, starting with $\mathbf{x}_0 = [1, 1, 1]^{\mathrm{T}}$ and performing 10 iterations.

1. $\mathbf{A} = \begin{bmatrix} 18 & 3 & -1 \\ 3 & 12 & 2 \\ -1 & 2 & 4 \end{bmatrix}$.

2. $\mathbf{A} = \begin{bmatrix} 20 & -2 & 1 \\ -2 & 3 & 4 \\ 1 & 4 & 0 \end{bmatrix}$.

3. $\mathbf{A} = \begin{bmatrix} 2 & -3 & 2 \\ -3 & 12 & 1 \\ 2 & 1 & 28 \end{bmatrix}$.

4. $\mathbf{A} = \begin{bmatrix} -31 & -1 & 2 \\ -1 & -10 & 4 \\ 2 & 4 & -2 \end{bmatrix}$.

In Exercises 5 and 6 use the power method to find approximations to the dominant eigenvalue λ_1, and the associated normalized eigenvector, starting with $\mathbf{x}_0 = [1, -1, 1]^{\mathrm{T}}$ and performing 10 iterations.

5. $\mathbf{A} = \begin{bmatrix} 26 & 3 & 1 \\ 3 & 20 & 2 \\ 1 & 2 & 1 \end{bmatrix}$.

6. $\mathbf{A} = \begin{bmatrix} 19 & 2 & 2 \\ 2 & 14 & 1 \\ 2 & 1 & 2 \end{bmatrix}$.

In Exercises 8 through 10 use the inverse power method to find approximations to the eigenvalue of smallest magnitude of the given matrix $\mathbf{A}$ and its associated eigenvector, starting with $\mathbf{x}_0 = [1, 1, 1]^{\mathrm{T}}$ and performing six iterations.

7. $\mathbf{A} = \begin{bmatrix} 6 & 1 & -4 \\ 1 & 4 & 0 \\ -1 & -1 & 3 \end{bmatrix}$.

8. $\mathbf{A} = \begin{bmatrix} 3 & 3 & -4 \\ 3 & 5 & 0 \\ -5 & -1 & 1 \end{bmatrix}$.

9. $\mathbf{A} = \begin{bmatrix} 2 & 5 & -2 \\ 4 & 2 & 4 \\ -3 & 1 & 0 \end{bmatrix}$.

10. $\mathbf{A} = \begin{bmatrix} -3 & 5 & -3 \\ 3 & 1 & 1 \\ -2 & 1 & 2 \end{bmatrix}$.

19.7 Numerical Solution of Differential Equations

Most differential equations have no known analytical solution, and even when one can be found it is often difficult to use. As a result, when solutions are required and an analytical solution either is not known or is inconvenient to use, it becomes necessary to use methods that produce a numerical solution directly. However, unlike the general analytical solution of an initial value problem that can be adapted to any appropriate initial conditions, a numerical solution is the solution of a specific

initial value problem, so the calculation must be repeated if the initial conditions are changed.

Many different techniques are available for the generation of a numerical solution of an initial value problem, the most powerful of which are implemented in the various numerical analysis software packages that are available. These include extrapolation methods, codes based on a family of Adams–Moulton methods, and others that use predictor–corrector methods with an Adams–Basforth method as the predictor and an Adams–Moulton method as a corrector. References for these methods are given later. In this section attention will be confined to the popular family of Runge–Kutta methods.

Predictor–corrector methods first use an explicit formula and previously computed solutions to predict a new solution. This prediction is then refined by using it in an implicit corrector formula. The Runge–Kutta methods are *one-step* methods, in the sense that the solution of a differential equation at the next step is determined solely by the solution at the previous step.

To illustrate how numerical solutions can be obtained by Runge–Kutta type methods, and to show the varying degrees of accuracy that can be attained by different approaches, a few of the simpler methods of this type will be described.

Euler's Method

The basis of this method has already been encountered in Section 5.3 when considering the **direction field** that can be associated with the first order differential equation

$$\frac{dy}{dx} = f(x, y). \tag{51}$$

Preparatory to developing Euler's method let us first recall the definition of the direction field associated with (51). At any point (x_0, y_0) in the (x, y)-plane at which $f(x, y)$ is defined, (51) shows that the slope (gradient) of the solution curve through the point is $f(x_0, y_0)$. If a short line segment is drawn through the point (x_0, y_0), making an angle θ with the positive x-axis, where $\tan\theta = f(x_0, y_0)$, the line segment will be *tangent* to the solution curve through (x_0, y_0). This line segment will define a *direction* of change of the solution at the point (x_0, y_0) if an arrow is added to the line segment indicating the sense in which y changes at that point as x increases. A repetition of this construction at a mesh of points over the region of the (x, y)-plane in which differential equation (51) is defined will then generate a **direction field** associated with the equation. Examples of direction fields have already been given in Chapter 5, and another for the linear differential equation

a typical direction field

$$\frac{dy}{dx} = \sin x - y$$

is shown in Fig. 19.11.

It is a short step from the notion of the direction field for differential equation (51) to **Euler's algorithm** for the solution of an initial value problem for the differential equation. An approximate numerical solution by Euler's method for the initial value problem

$$\frac{dy}{dx} = f(x, y), \quad \text{subject to the initial condition } y(x_0) = y_0, \tag{52}$$

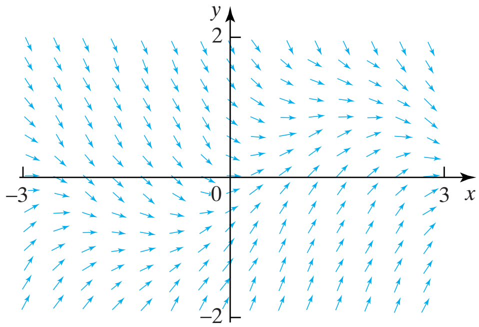


FIGURE 19.11 The direction field for $\frac{dy}{dx} = \sin x - y$.

is obtained as follows. A step size h in x is chosen, and the line segment through (x_0, y_0) is extended from x_0 to $x_0 + h$, and the y-coordinate $y_0 + \Delta y$ of the end point of the line segment is taken as the approximation to y at $x_0 + h$.

An increase in x of h from x_0 will cause the point on the tangent line approximation to the solution curve through (x_0, y_0) to increase from y_0 to $y_0 + \Delta y$, where $\Delta y = h \tan\theta$, but $\tan\theta = f(x_0, y_0)$, so $\Delta y = hf(x_0, y_0)$. It then follows that if P is the point $(x_0 + h, y_1)$ on the tangent line approximation (cf Fig. 19.4),

$$y_1 = y_0 + hf(x_0, y_0). \tag{53}$$

A repetition of this process produces a sequence of points $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n), \ldots$, where $x_n = x_0 + nh$ and $n = 0, 1, 2, \ldots$. When these points are joined by straight line segments, a polygonal line approximation to the solution of the initial value problem in (52) is generated, called an **Euler polygonal approximation** to the solution. The algorithm for generating such an approximate solution is easily seen to be as follows.

finding an approximate solution by the Euler method

The Euler algorithm

The approximate numerical solution of the initial value problem

$$\frac{dy}{dx} = f(x, y) \quad \text{subject to the initial condition } y(x_0) = y_0$$

generated by the Euler method with step size h is obtained from the algorithm

$$y_n = y_{n-1} + hf(x_{n-1}, y_{n-1}) \quad \text{for } n = 1, 2, \ldots,$$

where $x_n = x_0 + nh$.

This is the simplest example of a one-step method, and an obvious modification involves varying the step size from point to point, reducing it when the solution changes rapidly and lengthening it when it changes slowly. However, it is not possible to make such changes in a systematic manner without first having a way of estimating the error. This is usually done by comparing the result at each step to the result obtained by using a formula of higher order.

EXAMPLE 19.12 Use the Euler algorithm with a step size $h = 0.2$ to find an approximate solution of the linear first order initial value problem

$$\frac{dy}{dx} = \sin x - y \quad \text{with } y(0) = 1$$

in the interval $0 \le x \le 2$, and compare it with the exact solution

$$y = \frac{1}{2}(\sin x - \cos x) + \frac{3}{2}e^{-x}.$$

Solution This is an initial value problem for the differential equation whose direction field is shown in Fig. 19.11. Setting $h = 0.2$, $n = 10$, and $f(x, y) = \sin x - y$ in the Euler algorithm leads to the following results. The column y_{exact} contains the analytical solution.

n	x_n	y_n	$0.2f(x_n, y_n)$	$y_{n+1} = y_n + 0.2f(x_n, y_n)$	y_{exact}
0	0	1	−0.2	0.8	1
1	0.2	0.8	−0.1203	0.6797	0.8374
2	0.4	0.6797	−0.0581	0.6217	0.7397
3	0.6	0.6217	−0.0114	0.6103	0.6929
4	0.8	0.6103	0.0214	0.6317	0.6843
5	1	0.6317	0.0420	0.6736	0.7024
6	1.2	0.6736	0.0517	0.7253	0.7366
7	1.4	0.7253	0.0520	0.7773	0.7776
8	1.6	0.7773	0.0444	0.8218	0.8172
9	1.8	0.8218	0.0304	0.8522	0.8485
10	2	0.8522	0.0114	0.8636	0.8657

The error between y_{n+1} and y_{exact} can be reduced, but not eliminated, by choosing a smaller step size, though for significantly greater accuracy it is necessary to make use of a different method. ■

Modified Euler's Method

A source of error in Euler's method is its failure to take account of the curvature of the solution curve at a point (x_i, y_i) when using the tangent line approximation to the curve to estimate y_{i+1}. An improvement can be obtained by using a two-stage process to arrive at a modified gradient $\tilde{f}(x_i, y_i)$ that can be used in Euler's method in place of $f(x_i, y_i)$.

The first step when finding the modified gradient involves computing the gradient $f(x_i, y_i)$ and then using it in Euler's method to compute the gradient $f(x_{i+1}, y_{i+1})$ at the point (x_{i+1}, y_{i+1}). The second and final step involves averaging these two gradients, to obtain the new gradient

$$\tilde{f}(x_i, y_i) = \frac{1}{2}\{f(x_i, y_i) + f(x_{i+1}, y_{i+1})\}, \tag{54}$$

and then using $\tilde{f}(x_i, y_i)$ in place of $f(x_i, y_i)$ in Euler's method at (x_i, y_i) to find an improved estimate $\tilde{y}_{i+1}$ at the point (x_{i+1}, y_{i+1}). This way of computing the

gradient is known as **Heun's method**, and it takes some account of the curvature of the solution curve at (x_i, y_i). The following is an algorithm for the modified Euler method.

The modified Euler algorithm

The approximate numerical solution of the initial value problem

$$\frac{dy}{dx} = f(x, y) \quad \text{subject to the initial condition } y(x_0) = y_0$$

generated by the modified Euler method with step size h is obtained from the algorithm

finding an approximate solution by the modified Euler method

$$y_{n+1} = y_n + \frac{1}{2}h[f(x_n, y_n) + f(x_n + h, y_n + hf(x_n, y_n))] \quad \text{for } n = 1, 2, \ldots,$$

where $x_n = x_0 + nh$.

EXAMPLE 19.13

Repeat Example 19.12 using the modified Euler method with $n = 10$ and $h = 0.2$, and compare the results obtained with both the Euler method and the exact solution.

Solution The results of the calculations together with the comparisons are shown in the following table, in which results obtained using Euler's method are denoted by $y_n^{(e)}$, results obtained using Euler's modified method are denoted by $y_n^{(mod)}$, and the analytical result is denoted by y_{exact}. As the calculations are straightforward, the details have been omitted.

n	0	1	2	3	4	5	6	7	8	9	10
x_n	0	0.2	0.4	0.6	0.8	1	1.2	1.4	1.6	1.8	2
$y_n^{(e)}$	1	0.8	0.6797	0.6217	0.6103	0.6317	0.6736	0.7253	0.7773	0.8212	0.8522
$y_n^{(mod)}$	1	0.8399	0.7435	0.6973	0.6887	0.7063	0.7397	0.7796	0.8181	0.8482	0.8643
y_{exact}	1	0.8374	0.7397	0.6929	0.6843	0.7024	0.7366	0.7776	0.8172	0.8485	0.8657

A comparison of the results in last three rows of the table shows the improvement in accuracy obtained when the modified Euler method is used. ■

Euler's method is effectively a Taylor series expansion of the solution $y(x)$, in which $y(x_n + h)$ is predicted from $y(x_n)$ using only the first two terms of the Taylor series expansion of $y(x)$ about the point x_n. An often-used general purpose numerical method for the integration of initial value problems for first order differential equations is the Runge–Kutta fourth order method.

There are several families of four-stage, fourth order Runge–Kutta formulas in which the error after a step size h is of the order h^5, but as their derivation involves tedious algebra we will simply describe the most familiar one. However, before quoting this method, we first demonstrate the general approach to the derivation of Runge–Kutta methods by finding the modified Euler method.

In essence, all Runge–Kutta methods are one-step methods that can be considered to be of the form

$$y_{i+1} = y_i + hF(x_i, y_i, h), \tag{55}$$

where $F(x_i, y_i, h)$ represents some form of *averaged* value of $f(x, y)$ over the interval $x_i \le x \le x_{i+1}$. All of these methods can be obtained by adopting a particular form of F that contains some undetermined constants, and then finding the equations determining the constants by requiring that F agree with the Taylor series expansion of f up to a certain power of h.

a Runge–Kutta type derivation of the modified Euler method

In the case where F contains terms up to order h, so the error at each step will be of order h^2, using the chain rule and the fact that $f(x, y) = f(x, y(x))$, the function F in (55) is approximated by the truncated Taylor series expansion

$$F(x, y, h) = f(x, y) + \frac{1}{2}h\left\{\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}\right\},$$

but $dy/dx = f(x, y)$, so

$$F(x, y, h) = f(x, y) + \frac{1}{2}h\{f_x(x, y) + f_y(x, y)f(x, y)\}. \tag{56}$$

We now seek a representation of the function F of the form

$$F(x, y, h) = w_1 f(x, y) + w_2 f(x + w_3 h, y + w_4 h f(x, y)), \tag{57}$$

where as yet the constants w_1 to w_4 are unknown. Expanding $f(x + w_3 h, y + w_4 h f(x, y))$ about the point (x, y) as a two-variable Taylor series with a remainder after the first derivative terms gives

$$\begin{aligned} f(x + w_3 h, y + w_4 h(f(x, y)) = {} & f(x, y) + w_3 h f_x(x, y) \\ & + w_4 h f_y(x, y) f(x, y) + R(h), \end{aligned} \tag{58}$$

where the error term $R(h)$ is of order h^2.

Substituting (58) into (57) and combining terms gives

$$F(x, y, h) = (w_1 + w_2) f(x, y) + h(w_2 w_3 f_x(x, y) + w_2 w_4 f_y(x, y) f(x, y)). \tag{59}$$

If (57) and (59) are required to agree up to terms in h, by equating terms with corresponding powers of h we find that

$$w_1 + w_2 = 1, \quad w_2 w_3 = \frac{1}{2}, \quad \text{and} \quad w_2 w_4 = \frac{1}{2}.$$

These three equations relate the four arbitrary constants w_1 to w_4, so if one of these constants, say w_2, is assigned arbitrarily, the others will be determined in terms of w_2. From (57) we then have

$$F(x, y, h) = (1 - w_2) f(x, y) + w_2 f\left(x + \frac{1}{2}h/w_2, y + \frac{1}{2}hf(x, y)/w_2\right). \tag{60}$$

Making the choice $w_2 = \frac{1}{2}$ in (60), and using it in (55), gives the modified Euler method

$$y_{i+1} = y_i + \frac{1}{2}h\{f(x_i, y_i) + f(x_i + h, y_i + h\ f(x_i, y_i))\}. \tag{61}$$

CARL DAVID TOLME RUNGE (1856–1927)
A German mathematician who was Professor of Applied Mathematics at Göttingen. His interests were in the numerical solution of differential equations, and his approach was applied by **Wilhelm Kutta (1867–1944)**, a German aerodynamicist who used Runge's work in the study of fluid mechanics.

The fourth order Runge–Kutta method for a first order differential equation

The approximate numerical solution of the initial value problem

$$\frac{dy}{dx} = f(x, y) \text{ subject to the initial condition } y(x_0) = y_0$$

with step length h is obtained from the following fourth order Runge–Kutta algorithm, with $x_n = x_0 + nh$ and $y_n = y(x_n)$.

STEP 1 Calculate

$$\begin{aligned}
k_{1n} &= hf(x_n, y_n)\\
k_{2n} &= hf\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_{1n}\right)\\
k_{3n} &= hf\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_{2n}\right)\\
k_{4n} &= hf(x_n + h, y_n + k_{3n}).
\end{aligned}$$

STEP 2 Calculate

$$d_n = \frac{1}{6}(k_{1n} + 2k_{2n} + 2k_{3n} + k_{4n}).$$

STEP 3 The numerical approximation y_{n+1} of the solution $y = y(x_{n+1})$ is given by

$$y_{n+1} = y_n + d_n,$$

for $n = 1, 2, \ldots.$

EXAMPLE 19.14

Use the fourth order Runge–Kutta algorithm with a step size $h = 0.2$ to solve the initial value problem

$$\frac{dy}{dx} + 2y = \sin 3x \quad \text{with } y(0) = 1$$

in the interval $0 \le x \le 2.4$. Compare the results obtained with the results found by

the modified Euler method and the analytical solution

$$y = \frac{1}{13}[9\cos x - 2\sin x + 4\sin 2x\cos x - 12\cos^3 x + 16e^{-2x}].$$

Solution In the following calculations $f(x, y) = \sin 3x - 2y$, the step length $h = 0.2$, so as the solution is required in the interval $0 \le x \le 2.4$ it follows that $n = 0, 1, \ldots, 12$. The details of the intermediate calculations for $x = 0, 0.2, 0.4$, and 0.6 are listed in the first of the following tables. Under the heading y_{rk}, the second table lists all of the results obtained by the Runge–Kutta algorithm up to $x = 2.4$, and for purposes of comparison the columns with headings y_{mod} and y_{exact} show the results obtained by using the modified Euler method and the analytical solution, respectively.

Detailed Calculations for $x = 0, 0.2$, and 0.4

n	x_n	y_n	$f(x_n, y_n)$	k_{1n}	k_{2n}	k_{3n}	k_{4n}	y_{n+1}
0	0	1	−2	−0.4	−0.2609	−0.28872	−0.17158	0.72153
1	0.2	0.72153	−0.87842	−0.17568	−0.09681	−0.11258	−0.05717	0.61292
2	0.4	0.61292	−0.29380	−0.05876	−0.03392	−0.03889	−0.03484	0.57305
3	0.6	0.57305	—	—	—	—	—	—

Comparison of Results in the Interval $0 \le x \le 2.4$

n	x_n	y_{rk}	y_{mod}	y_{exact}	n	x_n	y_{rk}	y_{mod}	y_{exact}
0	0	1.0	1.0	1.0	7	1.4	0.05390	0.05090	0.05389
1	0.2	0.72153	0.73646	0.72142	8	1.6	−0.12324	−0.11730	−0.12328
2	0.4	0.61292	0.62788	0.61279	9	1.8	−0.23165	−0.21681	−0.23173
3	0.6	0.57305	0.58026	0.57295	10	2.0	−0.24192	−0.22174	−0.24202
4	0.8	0.52262	0.52056	0.52257	11	2.2	−0.15615	−0.13639	−0.15624
5	1.0	0.41675	0.40862	0.41674	12	2.4	−0.00809	−0.00531	−0.00816
6	1.2	0.25051	0.24208	0.25052	—	—	—	—	—

The fourth order Runge–Kutta algorithm is easily adapted to solve two simultaneous first order differential equations or, as a special case, a single second order differential equation as follows.

The fourth order Runge–Kutta algorithm for two first order simultaneous equations

The approximate numerical solution of the initial value problem for the simultaneous first order initial value problem

$$\frac{dy}{dx} = f(x, y, z) \quad \text{and} \quad \frac{dz}{dx} = g(x, y, z)$$

subject to the initial conditions

$$y(x_0) = y_0 \quad \text{and} \quad z(x_0) = z_0$$

generated by the fourth order Runge–Kutta method with step size h is obtained from the following algorithm in which $x_n = x_0 + nh$, $y_n = y(x_n)$, and $z_n = z(x_n)$.

STEP 1 Calculate in the following order

$$
\begin{aligned}
k_{1n} &= hf(x_n, y_n, z_n) & K_{1n} &= hg(x_n, y_n, z_n)\\
k_{2n} &= hf\left(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}k_{1n}, z_n + \tfrac{1}{2}K_{1n}\right) & K_{2n} &= hg\left(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}k_{1n}, z_n + \tfrac{1}{2}K_{1n}\right)\\
k_{3n} &= hf\left(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}k_{2n}, z_n + \tfrac{1}{2}K_{2n}\right) & K_{3n} &= hg\left(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}k_{2n}, z_n + \tfrac{1}{2}K_{2n}\right)\\
k_{4n} &= hf(x_n + h, y_n + k_{3n}, z_n + K_{3n}) & K_{4n} &= hg(x_n + h, y_n + k_{3n}, z_n + K_{3n})\,.
\end{aligned}
$$

STEP 2 Calculate

$$d_n = \frac{1}{6}(k_{1n} + 2k_{2n} + 2k_{3n} + k_{4n}) \quad \text{and}$$

$$D_n = \frac{1}{6}(K_{1n} + 2K_{2n} + 2K_{3n} + K_{4n}).$$

STEP 3 The numerical approximations of the solutions $y = y(x_{n+1})$ and $z = z(x_{n+1})$ are given by

$$y_{n+1} = y_n + d_n \quad \text{and} \quad z_{n+1} = z_n + D_n,$$

for $n = 1, 2, \ldots$.

---

adapting the Runge–Kutta method to solve second order equations

This fourth order Runge–Kutta algorithm with step size h is easily modified to find the solution of the following initial value problem for the single second order differential equation written in the standard form

$$\frac{d^2y}{dx^2} = g\left(x, y, \frac{dy}{dx}\right) \quad \text{with } y(x_0) = y_0 \text{ and } z(x_0) = z_0. \tag{62}$$

All that is necessary is to reduce the second order equation to a system of two simultaneous first order equations by setting

$$\frac{dy}{dx} = z \quad \text{and} \quad \frac{dz}{dx} = g(x, y, z) \tag{63}$$

in the preceding fourth order Runge–Kutta algorithm, and then to use the initial conditions

$$y(x_0) = y_0 \quad \text{and} \quad z(x_0) = y'(x_0) = z_0. \tag{64}$$

EXAMPLE 19.15

Use the fourth order Runge–Kutta algorithm with step length 0.1 to find a numerical approximation to the solution of the initial value problem for the Hermite equation

$$y'' - 2xy' + 8y = 0 \quad \text{with } y(0) = 12 \quad \text{and} \quad y'(0) = 0$$

in the interval $0 \le x \le 1$. Compare the results of the calculations with the analytical solution $y(x) = 16x^4 - 48x^2 + 12$.

Solution This is the Hermite equation with $n = 4$, and it has the analytical solution $H_4(x) = 16x^4 - 48x^2 + 12$. Using (62) and (63) we set $z = dy/dx$ and $g(x, y, z) = 2xz - 8y$, and use the step size $h = 0.1$. The initial conditions are imposed at the origin, so $x_0 = 0$, $y(x_0) = 12$, and $z(x_0) = y'(x_0) = 0$, corresponding to $y_0 = 12$ and $z_0 = 0$. The details of the intermediate calculations for $x = 0$ and 0.1 are set out below; the table that follows lists the results for the interval $0 \leq x \leq 1$, with the second order Runge–Kutta solution denoted by y_{rk} and the analytical solution by y_{exact}.

$\underline{x_0 = 0}$

$f(x_0, y_0, z_0) = 0$, $g(x_0, y_0, z_0) = -96$, $k_1 = 0$, $K_1 = -9.6$, $k_2 = -0.48$
$K_2 = -9.648$, $k_3 = -0.4824$, $K_3 = -9.45624$, $k_4 = -0.945624$
$K_4 = -9.403205$, $d = -0.478404$, $D = -9.535281$ so that
$y_1 = 11.521596$ and $z_1 = -9.535281$, where $z_1 = y'(x_1)$.

$\underline{x_1 = 0.2}$

$f(x_1, y_1, z_1) = -9.535281$, $g(x_1, y_1, z_1) = -94.079824$, $k_1 = -0.953528$,
$K_1 = -9.407982$, $k_2 = -1.423927$, $K_2 = -9.263044$, $k_3 = -1.416680$,
$K_3 = -9.072710$, $k_4 = -1.860799$, $K_4 = -8.828252$, $d = -1.415924$,
$D = -9.151290$ so that $y_2 = 10.105672$ and
$z_2 = -18.686571$, where $z_2 = y'(x_2)$.

Comparison of Solutions for $0 \leq x \leq 1$

n	x_n	y_{rk}	y_{exact}	n	x_n	y_{rk}	y_{exact}
0	0	12	12	6	0.6	−3.205311	−3.2064
1	0.1	11.521596	11.5216	7	0.7	−7.676938	−7.6784
2	0.2	10.105672	10.1056	8	0.8	−12.164555	−12.1664
3	0.3	7.809827	7.8096	9	0.9	−16.380188	−16.3824
4	0.4	4.730055	4.7296	10	1.0	−19.997470	−20.0
5	0.5	1.000747	1.0	—	—	—	—

When the solution of a differential equation changes rapidly in some intervals, and slowly in others, it becomes necessary to vary the step size as the calculation progresses if accuracy is to be maintained. The **F(4,5) Runge–Kutta–Fehlberg algorithm**, based on a form of the fourth order Runge–Kutta scheme, is implemented in many numerical analysis software programs that are readily available, and it determines the step size at each stage of the calculation. The increase in complexity of the calculation is indicated by the fact that the **F(4,5) algorithm** uses six stages in the calculation in place of the four used by the classical fourth order Runge–Kutta algorithm. As the calculation proceeds, numerical estimates of the solution after a given step size h are made using a form of the fourth order Runge–Kutta method and an efficient fifth order formula. The difference of these two estimates is compared with a preassigned tolerance, and the result is then used to either reduce or increase the step size until the difference lies within the required

the F(4,5) adaptive step size algorithm

tolerance. The resulting step size is then used to advance the calculation to the next stage.

More detailed information about the numerical integration schemes for ordinary differential equations can be found in references [2.19], [2.20], and [3.20] through [3.26].

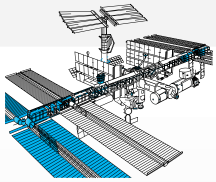
Summary

Of the many methods available for the numerical integration of ordinary differential equations, at an elementary level only the Euler and modified Euler methods have been described. For greater accuracy the classical fourth order Runge–Kutta algorithm, which belongs to a family of similar algorithms, was presented without derivation, though the form of argument used was illustrated by deriving the modified Euler method. Finally, the important adaptive F(4,5) Runge–Kutta–Fehlberg algorithm was mentioned that adjusts the step size automatically as the calculation progresses in order to preserve a preassigned accuracy.

EXERCISES 19.7

Solve the following initial value problems by computer using the fourth order Runge–Kutta algorithm.

1. $y' = (3x^2 + y^2)^{1/2} - y$ with $y(2) = 0$ and $h = 0.2$ over the interval $2 \le x \le 3$.
2. $y' = xy/(x^2 + y^2)^{1/2}$ with $y(1) = 1$ and $h = 0.2$ over the interval $1 \le x \le 2$.
3. $y' = (x^2 + y^2)^{1/2} - xy$ with $y(1) = 2$ and $h = 0.2$ over the interval $1 \le x \le 2$.
4. $y' = \frac{1}{2}(x^2 + 2y^2) - xy$ with $y(1) = 0$ and $h = 0.1$ over the interval $1 \le x \le 1.5$.
5. $y' = \cos(2x + y) - 3y$ with $y(1) = 1$ and $h = 0.2$ over the interval $1 \le x \le 2$.
6. $y' = \sin(x + y) - 2y$ with $y(0) = 1$ and $h = 0.2$ over the interval $0 \le x \le 1$.
7. $y'' - xyy' + 2y = 0$ with $y(0) = 2,\ y'(0) = -1$, and $h = 0.1$ over the interval $0 \le x \le 0.5$.
8. $y'' + (3 + x)y' + y^2 = 0$ with $y(1) = 1,\ y'(1) = 2$, and $h = 0.1$ over the interval $1 \le x \le 1.5$.
9. $y'' + (1 + \sin 2x)y' + 3y = 0$ with $y(0) = 1,\ y'(0) = 1$, and $h = 0.1$ over the interval $0 \le x \le 0.5$.
10. $y'' + (1 + y^2)^{1/2}y' + y = 0$ with $y(2) = 0,\ y'(2) = 1$, and $h = 0.1$ over the interval $2 \le x \le 2.5$.
11. $y'' + 2y' - y^2 = 0$ with $y(0) = 2,\ y'(0) = 1$, and $h = 0.2$ over the interval $0 \le x \le 1$.
12. $y'' - xy' - y^2 = 0$ with $y(0) = -1,\ y'(0) = 2$, and $h = 0.2$ over the interval $0 \le x \le 1$.
13. $y'' + yy' - 3y = 0$ with $y(1) = 1,\ y'(1) = 1$, and $h = 0.2$ over the interval $1 \le x \le 2$.
14. $y'' + x^2 \sin y' - 2y = 0$ with $y(1) = 0,\ y'(1) = -1$, and $h = 0.2$ over the interval $1 \le x \le 2$.
15. $y'' - xy' - y^2 = 2x$ with $y(0) = -2,\ y'(0) = 1$, and $h = 0.2$ over the interval $0 \le x \le 1$.
16. $y'' + 2yy' - 3y = 1 - x^2$ with $y(0) = 3,\ y'(0) = 2$, and $h = 0.2$ over the interval $0 \le x \le 1$.
17. $\frac{dx}{dt} = tx + (x + y)y$ and $\frac{dy}{dt} = ty - (x + y)x$ with $x(0) = 1,\ y(0) = 0$, and $h = 0.2$ over the interval $0 \le t \le 1$.
18. $\frac{dx}{dt} = (1 + t)y^2 - 2x$ and $\frac{dy}{dt} = y^2 + tx$ with $x(0) = -1$, $y(0) = -3$, and $h = 0.2$ over the interval $0 \le t \le 1$.
19. $\frac{dx}{dt} = \sin(x + 4y)$ and $\frac{dy}{dt} = 2\cos(x - 3y)$ with $x(0) = 1$, $y(0) = 1$, and $h = 0.2$ over the interval $0 \le t \le 1$.
20. $\frac{dx}{dt} = \sin x + 4\cos y$ and $\frac{dy}{dt} = \sin y - 3\sin x$ with $x(0) = 1,\ y(0) = -2$, and $h = 0.2$ over the interval $0 \le t \le 1$.



CHAPTER 19
TECHNOLOGY PROJECTS

Project 1

Spline Function Approximation

This project uses a spline function computer package to generate a natural spline approximation to a given data set. The data provided can be considered to be the scaled set of nine points through which the profile of the side elevation of a yacht hull complete with its keel must pass.

1. Make and plot a natural cubic spline function approximation to the following set of data points, where in each number pair the first number represents the x-coordinate and the second the y-coordinate:

$$(0, 0), (4.5, -2.3), (10, -3.7), (12.3, -6.8),$$
$$(16.7, -6.8), (18.4, -3.4), (21.2, -2.3), (23, 0).$$

2. Design a different profile of your own involving at least nine number pairs. Construct and plot a corresponding spline function approximation, and compare the result with the original profile. If necessary, reposition the data points to make the approximation a better fit.

Project 2

Newton's Method

The purpose of this project is to construct a procedure for Newton's method, and then to use it to determine the zeros of two expressions involving Bessel functions.

1. Plot $f(x) = J_2(x)$ for $0 \le x \le 35$ and use the graph to determine the approximate zeros of $J_2(x)$ in this interval, the first six of which are listed in Table 8.1. Construct a procedure for Newton's method involving 10 iterations and use it with the approximate values found from the graph to determine the zeros of $f(x)$ to 10 decimal places. Print out the values of these zeros together with the value of $f(x)$ at each zero to confirm the accuracy.
2. Repeat some of the previous calculations using poorer initial approximations to experience how sometimes the calculation does not converge to the expected zero and sometimes it diverges. Notice that this numerical method only works when $f'(x)$ can be found analytically.
3. The eigenvalues of a certain problem are determined by the zeros of the expression

$$J_0(x)J_1(1.5x) - J_0(1.5x)J_1(x) = 0.$$

Plot $f(x) = J_0(x)J_1(1.5x) - J_0(1.5x)J_1(x)$ in the interval $0 \le x \le 20$ and determine from the graph the approximate values of the first three positive zeros of $f(x)$. Use the procedure developed in part 1 with these approximate zeros to find their values to 10 decimal places.

Project 3

Modified Euler and Runge–Kutta Methods

The purpose of this project is to construct procedures for the modified Euler and the fourth order Runge–Kutta method and then to compare the results obtained when they are applied first to a simple linear initial value problem and then to a nonlinear initial value problem.

1. Construct a procedure for the modified Euler method derived in Section 19.7.
2. Construct a procedure for the fourth order Runge–Kutta method defined as follows:

Consider the differential equation $dy/dx = f(x, y)$, and let the initial condition at $x = x_0$ be $y(x_0) = y_0$. Let the step size be h and $y_1, y_2, \ldots, y_r$ be the approximations to $y(x)$ at the respective points $x_1 = x_0 + h, x_2 = x_0 + 2h, \ldots, x_r = x_0 + rh$. Then, for $n = 0, 1, \ldots,$ the values $y_1, y_2, \ldots$ are determined from the

algorithm

$$k_1 = hf(x_n, y_n)$$
$$k_2 = hf\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_1\right)$$
$$k_3 = hf\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_2\right)$$
$$k_4 = hf(x_n + h, y_n + k_3)$$
with $x_{n+1} = x_n + h$ and
$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4).$$

3. Apply both methods to the linear initial value problem $dy/dx = y$ with $y(0) = 1$ and $h = 0.1$. Print out the results for the interval $0 \le x \le 1$ and compare them with the exact solution $y(x) = e^x$.
4. Apply both methods to the nonlinear initial value problem

$$dy/dx = \sin(xy)\sin(3x), \quad \text{with} \quad y(0) = 1 \text{ and } h = 0.1,$$

and compare the results over the interval $0 \le x \le 2$.

Project 4

The Shooting Method

This project provides an introduction to the **shooting method** when used to solve a two-point boundary value problem for a linear second order differential equation. The underlying idea of the method can be likened to the problem of projecting a particle from a fixed point at different angles to the horizontal, and finding the angle of projection at which the particle attains a prescribed altitude when at a fixed horizontal distance from its point of origin.

Consider the two-point boundary value problem

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R(x),$$
$$\text{with } y(a) = k \text{ and } y(b) = K \quad (b > a),$$

where a, b, k, and K are given numbers.

Now consider two initial value problems with the different initial conditions

$$\text{(I)} \qquad y(a) = k \quad \text{and} \quad y'(a) = K_1$$

and

$$\text{(II)} \qquad y(a) = k \quad \text{and} \quad y'(a) = K_2,$$

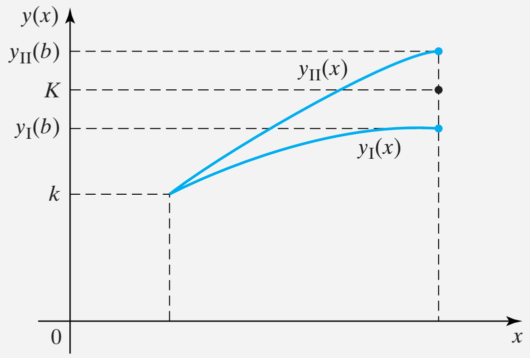


FIGURE 19.12 The two solutions $y_I(x)$ and $y_{II}(x)$.

where for the moment numbers $K_1 \ne K_2$ are specified arbitrarily. If the corresponding solutions are $y_I(x)$ and $y_{II}(x)$, their typical behavior is shown in Fig. 19.12, where the value $y(b) = K$ necessary to satisfy the original two-point boundary value problem is shown as the point (b, K).

Now set $y(x) = K_1 y_I(x) + K_2 y_{II}(x)$, with $K_1 + K_2 = 1$. Then substituting this result into the differential equation shows that it is a solution and, in addition, that $y(x)$ satisfies the boundary condition $y(a) = k$. Setting $x = b$ and $y(b) = K$ in $y(x)$ gives

$$K = K_1 y_I(b) + K_2 y_{II}(b),$$

so using the condition $K_1 + K_2 = 1$, solving for K_1 and K_2, and substituting the results into $y(x)$ shows that the solution of the two-point boundary value problem is given by

$$y(x) = \left[\frac{K - y_{II}(b)}{y_I(b) - y_{II}(b)}\right] y_I(x) + \left[\frac{y_I(b) - K}{y_I(b) - y_{II}(b)}\right] y_{II}(x).$$

Using the fourth order Runge-Kutta method to find $y_I(x)$ and $y_{II}(x)$, apply this method to the two-point boundary value problem

$$2x^2\frac{d^2y}{dx^2} - 7x\frac{dy}{dx} + 10y = 3x,$$
$$\text{with } y(1) = 1 \text{ and } y(2) = 4,$$

and find the solution for $1 \le x \le 2$ at step increments of 0.2. Compare your result with the analytical solution

$$y(x) = x + \frac{x^2(1 - \sqrt{x})}{2 - 2\sqrt{2}}, \quad \text{for } 1 \le x \le 2.$$

Project 5

Least Squares Fitting of Data

Instead of Lagrange or spline interpolation between known data points, it is sometimes better to fit an expression of the form

$$P(x) = a_0\varphi_0(x) + a_1\varphi_1(x) + \cdots + \varphi_m(x),$$

where the $\varphi_0(x), \varphi_1(x), \ldots, \varphi_m(x)$ is some convenient set of functions. In the method of least squares, the function $P(x)$ is fitted to the set of data points $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$ by setting

$$S(a_0, a_1, \ldots, a_m) = \sum_{i=0}^{n}[P(x_j) - y_i]^2,$$

and requiring this sum of squares of errors between $P(x)$ at the points x_i and the corresponding numbers y_i to be minimized.

A typical case involves fitting a quadratic in x to the data points, so $\varphi_r(x) = x^r$ and

$$P(x) = a_0 + a_1 x + a_2 x^2.$$

The method of least squares then requires the sum $S(a_0, a_1, a_2)$ to be minimized, where

$$S(a_0, a_1, a_2) = \sum_{i=0}^{n}\left[a_0 + a_1 x_i + a_2 x_i^2 - y_i\right]^2.$$

Regarding the coefficients a_0, a_1, a_2 as parameters, the extremum of the square error S will be found by taking the coefficients to be the solutions of the three equations $\partial S/\partial a_j = 0$; that is, by finding a_0, a_1, and a_2 from the three linear nonhomogeneous equations

$$a_0 \sum_{r=0}^{n} x_r^j + a_1 \sum_{r=0}^{n} x_r^{j+1} + a_2 \sum_{r=0}^{n} x_r^{j+2} = \sum_{r=0}^{n} x_r^j y_r, \quad \text{for } j = 0, 1, 2.$$

Substituting the coefficients a_0, a_1, and a_2 into $P(x)$ then gives the required least squares fit.

(a) Define a function $f(x)$ that can reasonably be approximated by $P(x) = a_0 + a_1 x + a_2 x^2$ over an interval $x_0 \le x \le x_n$. For some arbitrary increasing set of points $x_0, x_1, \ldots, x_n$, typically with $n = 20$, set $y_j = f(x_j)$. Using the points $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$ as data points, make a least squares fit of $P(x)$. Plot $P(x)$ and the data points together to show the nature of fit that is obtained. Examine how changing the set of points $x_0, x_1, \ldots, x_n$ alters the nature of the fit.

(b) Extend the preceding analysis using $P(x) = a_0 = a_1 x + a_2 x^2 + a_3 x^3$. Repeat the calculations in (a), but this time using a function $f(x)$ that can reasonably be approximated by a cubic. Again plot $P(x)$ and the original set of data points together to show the nature of the fit. Again examine how changing the set of points $x_0, x_1, \ldots, x_n$ alters the nature of the fit.

ANSWERS

Exercise Set 1.1

1. Consider $a/\sqrt{b}+b/\sqrt{a}-\sqrt{a}-\sqrt{b}=[a-\sqrt{(ab)}]/\sqrt{b}+[b-\sqrt{(ab)}]/\sqrt{a}=(a-b)(\sqrt{a}-\sqrt{b})/\sqrt{(ab)}\geq 0$. Numerator and denominator have the same sign, so the result follows.
3. $P(n)$ is the stated proposition and $P(1)$ is true. $(1-r^n)/(1-r)+r^n=(1-r^{n+1})/(1-r)$ so $P(n)$ implies $P(n+1)$, but $P(1)$ is true so $P(n)$ is true for $n\geq 1$.
5. Use the same form of argument as in Example 1.1. A quick noninductive proof follows from Example 1.1 by replacing ax by $ax+\pi/2$.
7. $81+216x+216x^2+96x^3+16x^4$
9. $\frac{1}{9}-\frac{4}{27}x+\frac{4}{27}x^2-\frac{32}{243}x^3+\cdots,\ |x|<\frac{3}{2}$
11. $\frac{1}{2}-\frac{1}{8}x^2+\frac{3}{64}x^4-\frac{5}{256}x^6+\cdots,\ |x|<\sqrt{2}$

Exercise Set 1.2

1. $-\frac{1}{2}\pm i\frac{\sqrt{3}}{2}$
3. $-\frac{1}{2}\pm i\frac{\sqrt{23}}{2}$
5. $-\frac{1}{2}\pm i\frac{\sqrt{3}}{6}$
7. $-\frac{1}{4}\pm i\frac{\sqrt{31}}{4}$
9. $a=5,\ b=-40$
11. $\sqrt{10},\ 4-i,\ -7-3i,\ 8-i,\ -30-45i,\ \sqrt{65}/5,\ \frac{1}{5}+i\frac{4}{15}$

Exercise Set 1.3

1. $u+v=3+i,\ u-v=1+5i$
3. $u+v=-6-4i,\ u-v=4i$
5. $u+v=-1+8i,\ u-v=7+4i$
7. $u+v=-8-8i,\ u-v=12i$

Exercise Set 1.4

1. Straightforward
3. Expand the left side of the identity $(\cos\theta+i\sin\theta)^5=\cos 5\theta+i\sin 5\theta$ and then equate real and imaginary parts to obtain $\cos 5\theta=\cos^5\theta-10\cos^3\theta\sin^2\theta+5\cos\theta\sin^4\theta$ and $\sin 5\theta=5\cos^4\theta\sin\theta-10\cos^2\theta\sin^3\theta+\sin^5\theta$.
5. Straightforward
7. $z^n+1/z^n=\exp(in\theta)+\exp(-in\theta)=2\cos n\theta$, so $\cos n\theta=\frac{1}{2}(z^n+1/z^n)$ and, similarly, $\sin n\theta=\frac{1}{2i}(z^n-1/z^n)$, and with $n=1$, $\cos\theta=\frac{1}{2}(z+1/z)$ and $\sin\theta=\frac{1}{2i}(z-1/z)$. Thus $\cos^3\theta\sin^3\theta=(1/2)^3(z+1/z)^3(1/2i)^3(z-1/z)^3$. Expanding, grouping terms, and using the above results gives $\cos^3\theta\sin^3\theta=\frac{3}{32}\sin 2\theta-\frac{1}{32}\sin 6\theta$.
9. Proceed as in Exercise 7.
11. Proceed as in Exercise 7.
13. $8\sqrt{2}\exp(i\pi/12),\ \sqrt{2}/4\exp(7i\pi/12),\ 128\exp(-2\pi i/3)$
15. $24\exp(-i\pi/3),\ 2/3\exp(i\pi/3),\ \sqrt{2}/32\exp(-i\pi/4)$
17. $64,\ \pi/2$
19. Multiply numerator and denominator on the right of Exercise 18 by $e^{i\theta/2}$ to obtain $\sum_{k=1}^{n}\exp(ik\theta)=\frac{\exp[i(n+\frac{1}{2})\theta]-\exp(\frac{1}{2}i\theta)}{\exp(\frac{1}{2}i\theta)-\exp(-\frac{1}{2}i\theta)}$. The Lagrange identity follows by equating the real parts of this identity.

Exercise Set 1.5

1. $\pm\left\{\left(\frac{\sqrt{2}-1}{2}\right)^{1/2}+i\left(\frac{\sqrt{2}+1}{2}\right)^{1/2}\right\}$
3. $\pm\frac{1}{\sqrt{2}}(1+i)$
5. $\pm\left\{\left(\frac{\sqrt{13}+2}{2}\right)^{1/2}-i\left(\frac{\sqrt{13}-2}{2}\right)^{1/2}\right\}$
7. $\pm(1/\sqrt{2})(3-i)$
9. $2^{1/3}\exp(\pi i/9),\ 2^{1/3}\exp(7\pi i/9),\ 2^{1/3}\exp(13\pi i/9)$
11. $-(1/\sqrt{2})(1+i),\ (1/\sqrt{2})(1-i),\ (1/\sqrt{2})(-1+i),\ (1/\sqrt{2})(1+i)$
13. $i,\ -(1/2)(\sqrt{3}+i),\ (1/2)(\sqrt{3}-i)$
15. $0,\ [(\sqrt{2}+1)/2]^{1/2}-i[(\sqrt{2}-1)/2]^{1/2},\ -[(\sqrt{2}+1)/2]^{1/2}+i[(\sqrt{2}-1)]^{1/2}$
17. ω may be any nth root of unity. Choose $\omega=\exp(2\pi i/n)$ and substitute for ω. The first result

follows by equating the real parts of the expression and the second by equating the imaginary parts.

19. $1, 2-3i, 2+3i$

21. The polynomial has complex coefficients, so its roots do not occur in complex conjugate pairs. $z_\pm = \pm[(1/\sqrt{2}+1/2)^{1/2} - i(1/\sqrt{2}-1/2)^{1/2}]$

Exercise Set 1.6

1. $\dfrac{5}{3}\dfrac{1}{2x+1} + \dfrac{2}{3}\dfrac{1}{x+2}$

3. $\dfrac{29}{2x+5} - \dfrac{13}{x+2}$

5. $\dfrac{1}{x+2} - \dfrac{1}{(x+2)^2}$

7. $1 - \dfrac{4}{x+2} + \dfrac{5}{(x+2)^2}$

9. $(x+2)^2+1$

11. $2(x+3/4)^2 - 57/8$

13. $9(x-1/9)^2+17/9$

Exercise Set 1.7

1. 18

3. 21

5. 0

7. 1

11. $x_1 = 10/23$, $x_2 = 15/23$, $x_3 = -6/23$

Exercise Set 1.13

1. In Theorem 1.10 set $n=3$ and make the identifications $x_1 = x$, $x_2 = y$, $x_3 = z$, $u_1 = r$, $u_2 = \theta$, $u_3 = z$, $X_1 = r\cos\theta$, $X_2 = r\sin\theta$, $X_3 = z$ and substitute into the theorem.

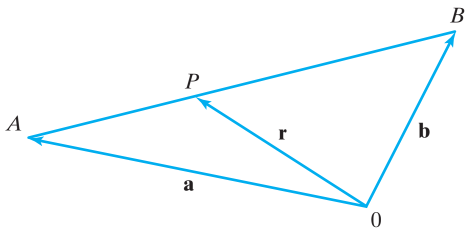
3. In Theorem 1.10 set $n=3$ and make the identifications $x_1 = x$, $x_2 = y$, $x_3 = z$, $u_1 = r$, $u_2 = \theta$, $u_3 = \phi$, $X_1 = r\sin\theta\cos\phi$, $X_2 = r\sin\theta\sin\phi$, $X_3 = r\cos\theta$ and substitute the results into the theorem.

Exercise Set 2.1

3. (a) $-(3/2)\mathbf{i} - \mathbf{j} - 3\mathbf{k}$ (b) $2\mathbf{i} - 9\mathbf{j} - 9\mathbf{k}$

5. $\underline{AB} = -\mathbf{i} - \mathbf{j} + 5\mathbf{k}$, unit vector is $(1/\sqrt{27})(-\mathbf{i} - \mathbf{j} + 5\mathbf{k})$

7. $\underline{AB} = \mathbf{b} - \mathbf{a}$, so the unit vector in this direction is $\hat{\mathbf{v}} = (\mathbf{b}-\mathbf{a})/|\mathbf{b}-\mathbf{a}|$. Divide AB into $m+n$ parts of length $|\mathbf{b}-\mathbf{a}|/(m+n)$, then AP $= m|\mathbf{b}-\mathbf{a}|/(m+n)$, so $\underline{AP} = \text{AP}\,\hat{\mathbf{v}} = m(\mathbf{b}-\mathbf{a})/(m+n)$. As $\underline{OP} = \underline{OA} + \underline{AP}$ we have $\mathbf{r} = \mathbf{a} + m(\mathbf{b}-\mathbf{a})/(m+n) = (n\mathbf{a}+m\mathbf{b})/(m+n)$.



9. Use the same form of argument as in Exercise 7 with M the mid-point of AC. Hence, show that $\underline{AM} = (\mathbf{c}-\mathbf{a})/2$, $\underline{OM} = \underline{OA} + \underline{AM} = (\mathbf{a}+\mathbf{c})/2$ and $\underline{MB} = \underline{OB} - \underline{OM} = \mathbf{b} - (\mathbf{a}+\mathbf{c})/2$. If P is 1/3 the distance along MB from M, $\underline{MP} = \underline{MB}/3$. Position vector $\underline{OP} = \underline{OM} + \underline{MP} = (1/3)(\mathbf{a}+\mathbf{b}+\mathbf{c})$. A similar argument yields the identical result using the other two mid-points of sides of the triangle, so the result is proved.

11. Let the forces along the x, y, and z axes be $\mathbf{F}_1, \mathbf{F}_2$, and $\mathbf{F}_3$. Then $\mathbf{F}_1 = 2\mathbf{i}$, $\mathbf{F}_2 = \mathbf{j}$, and $\mathbf{F}_3 = 4\mathbf{k}$, so $\mathbf{S} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = 2\mathbf{i} + \mathbf{j} + 4\mathbf{k}$, $\|\mathbf{S}\| = \sqrt{21}$, and $\hat{\mathbf{S}} = (1/\sqrt{21})(2\mathbf{i} + \mathbf{j} + 4\mathbf{k})$.

13. The standard form of the equation of L is $\frac{x+1/2}{3/2} = \frac{y+2/3}{4/3} = \frac{z-1/2}{1/4}$, so the position vector of a point on the line is $\mathbf{a} = -(1/2)\mathbf{i} - (2/3)\mathbf{j} + (1/2)\mathbf{k}$. A vector along the line is $\mathbf{b} = (3/2)\mathbf{i} + (4/3)\mathbf{j} - (1/4)\mathbf{k}$, so a unit vector along L is $\mathbf{b}/\|\mathbf{b}\|$ where $\|\mathbf{b}\| = \sqrt{589}/12$. To find the position vector of another point on L choose an arbitrary value for x and use it in the equation for L to find the corresponding values of y and z.

15. (a) As (3, 2, 4) lies on L_1 its position vector is $\mathbf{a} = 3\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$. As $(3,2,4)$ and $(2,1,6)$ also lie on L_1 a vector $\mathbf{b}$ along L_1 is $\mathbf{b} = (2\mathbf{i} + \mathbf{j} + 6\mathbf{k}) - (3\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}) = -\mathbf{i} - \mathbf{j} + 2\mathbf{k}$. (b) The line L_2 is also parallel to $\mathbf{b}$, but it passes through $\mathbf{a} = -2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$, so L_2 has the equation

$$\frac{x+2}{-1} = \frac{y-1}{-1} = \frac{z-2}{2}.$$

17. The position vector of a point on the line is $\mathbf{a} = 3\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ and a vector parallel to the line is $\mathbf{b} = 2\mathbf{i} + 3\mathbf{j} - 3\mathbf{k}$. If we set $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ the vector equation of the line $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ becomes $x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = 3\mathbf{i} + 2\mathbf{j} - 3\mathbf{k} + \lambda(2\mathbf{i} + 3\mathbf{j} - 3\mathbf{k})$, so the cartesian form of the equation is

$$\frac{x-3}{2} = \frac{y-2}{3} = \frac{z+3}{-3} = \lambda.$$

The coordinates of three arbitrary points on the line follow by assigning λ three different numerical values and then solving for x, y and z.

Exercise Set 2.2

1. (a) 2 (b) 4 (c) 0 **3.** (a) No (b) No (c) No (d) Yes

5. (a) 16 (b) -15 (c) 17 (d) 1

7. (a) $\cos\theta = (\sqrt{2}/3), \theta = 61.9^\circ$ (b) $\cos\theta = 6/7$, $\theta = 31^\circ$ (c) $\cos\theta = 8/\sqrt{154}, \theta = 49.9^\circ$

9. $F_C = \mathbf{F}\cdot\hat{\mathbf{n}} = \mathbf{F}\cdot(\mathbf{i}+\mathbf{j}+\mathbf{k})/\sqrt{3}$ so $F_C = 9/\sqrt{3}$

11. $\mathbf{a}\cdot\hat{\mathbf{b}} = -2/\sqrt{14}, \mathbf{b}\cdot\hat{\mathbf{a}} = -2/3$

13. **(a)** $l = m = n = 1/\sqrt{3}, \theta = 54.7^\circ$

(b) $l = 1/3, \theta = 70.5^\circ$, $m = -2/3$, $\theta = 131.8^\circ$, $n = 2/3, \theta = 48.2^\circ$

(c) $l = 4/\sqrt{29}$, $\theta = 42^\circ$, $m = -2/\sqrt{29}$, $\theta = 111.8^\circ, n = 3/\sqrt{29}, \theta = 56.1^\circ$

15. $\|\mathbf{a}\| = \sqrt{14}, \|\mathbf{b}\| = \sqrt{54}, \|\mathbf{a}+\mathbf{b}\| = \sqrt{118}, \sqrt{118} < \sqrt{14}+\sqrt{54}$

17. $2x - 3y + z = 3$

19. $2x + z = -1$

21. $\mathbf{r}\cdot\mathbf{n}/\|\mathbf{n}\|$ is the projection of the position vector of a point on the plane onto the unit normal to the plane, and so is the perpendicular distance of the plane from the origin. If $\mathbf{a}\cdot\mathbf{n} > 0$ the perpendicular distance of the plane from the origin is positive, so the plane then lies on the side of O toward which $\mathbf{n}$ is directed, and conversely.

23. $\mathbf{n}_1 = \mathbf{i}+3\mathbf{j}+2\mathbf{k}, \mathbf{n}_2 = 2\mathbf{i}-5\mathbf{j}+\mathbf{k}$ so $\cos\theta = \mathbf{n}_1\cdot\mathbf{n}_2/\|\mathbf{n}_1\|\|\mathbf{n}_2\| = -11/(\sqrt{14}\sqrt{30}), \theta = 122^\circ$

25. Component of $\mathbf{a}$ in direction of $\mathbf{b}$ is $a_b = \mathbf{a}\cdot\hat{\mathbf{b}}$ so $\mathbf{a}_b = (\mathbf{a}\cdot\mathbf{b})\mathbf{b}/\|\mathbf{b}\|^2$, but $\mathbf{a} = \mathbf{a}_b + \mathbf{a}_p$, so $\mathbf{a}_p = \mathbf{a} - (\mathbf{a}\cdot\mathbf{b})\mathbf{b}/\|\mathbf{b}\|^2$

27. $W = \mathbf{F}\cdot\hat{\mathbf{a}}d = (\mathbf{F}\cdot\mathbf{a}/\|\mathbf{a}\|)d$

29. $\|\mathbf{a}\|^2 = 26$, $\|\mathbf{b}\|^2 = 14$, $|\mathbf{a}\cdot\mathbf{b}| = 5$, $\|\lambda\mathbf{a}+\mu\mathbf{b}\|^2 = 170$. $170 \le (4)\cdot(26) - (12)\cdot(5) + (9)\cdot(14) = 170$, so in this case the equality holds.

Exercise Set 2.3

1. $5\mathbf{i} - 14\mathbf{j} + \mathbf{k}$

3. $-18\mathbf{i} - 7\mathbf{j} + 21\mathbf{k}$

5. $2\mathbf{i} - 4\mathbf{k}$

7. $-5\mathbf{i} + 8\mathbf{j} - \mathbf{k}$

9. $-2\mathbf{i} - 11\mathbf{j} - 5\mathbf{k}$

11. $(\mathbf{b}+\mathbf{c})\times\mathbf{a} = -24\mathbf{i} - 12\mathbf{j} + 18\mathbf{k}$

13. $(\mathbf{b}+\mathbf{c})\times\mathbf{a} = -7\mathbf{j}$

15. $(-\mathbf{i} - 2\mathbf{j} + 5\mathbf{k})/\sqrt{30}$

17. $(-4\mathbf{i} + 3\mathbf{j} + \mathbf{k})/\sqrt{26}$

19. $(\mathbf{i} - \mathbf{j})/\sqrt{z}$

21. $3x + 3y - z = 10$

23. $4x + 2z = 10$

25. No

27. Yes

29. $\mathbf{N} = \alpha\mathbf{i} + \beta\mathbf{j} + \gamma\mathbf{k}, \mathbf{a} = \mathbf{i}+\mathbf{j}+3\mathbf{k}, \mathbf{b} = 3\mathbf{i}+2\mathbf{j}+\mathbf{k}$ $\mathbf{a}\cdot\mathbf{N} = 0$ gives $\alpha+\beta+3\gamma = 0$ and $\mathbf{b}\cdot\mathbf{N} = 0$ gives $3\alpha+2\beta+\gamma = 0$. Set $\alpha = c$ (arbitrary). Then $\beta = -(8/5)c$ and $\gamma = (1/5)c$, so $\mathbf{N} = c(\mathbf{i} - (8/5)\mathbf{j} + (1/5)\mathbf{k})$ and $\hat{\mathbf{N}} = (5\mathbf{i} - 8\mathbf{j} + \mathbf{k})/\sqrt{90}$. Next $\mathbf{a}\times\mathbf{b} = -5\mathbf{i} + 8\mathbf{j} - \mathbf{k}$, so $\hat{\mathbf{n}} = (-5\mathbf{i} + 8\mathbf{j} - \mathbf{k})/\sqrt{90}$, showing that $\hat{\mathbf{N}} = -\hat{\mathbf{n}}$. The difference in sign is due to the fact that $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{N}$ do not necessarily form a right-handed set of vectors.

Exercise Set 2.4

1. $\mathbf{a}.(\mathbf{b}\times\mathbf{c}) = -15, V = 15$

3. $\mathbf{a}.(\mathbf{b}\times\mathbf{c}) = 25, V = 25$

5. Yes

7. No

9. Yes

11. $[\mathbf{a}, \mathbf{b}, \mathbf{c}] = -10$

13. $[\mathbf{a}, \mathbf{b}, \mathbf{c}] = 0$

15. $[\lambda\mathbf{a}+\mu\mathbf{b}, \mathbf{c}, \mathbf{d}] = (\lambda\mathbf{a}+\mu\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = \lambda\mathbf{a}\cdot(\mathbf{c}\times\mathbf{d}) + \mu\mathbf{b}\cdot(\mathbf{c}\times\mathbf{d}) = \lambda[\mathbf{a}, \mathbf{c}, \mathbf{d}] + \mu[\mathbf{b}, \mathbf{c}, \mathbf{d}]$

17. $7x + 2y - 4z = 2, \hat{\mathbf{n}} = (7\mathbf{i} + 2\mathbf{j} - 4\mathbf{k})/\sqrt{69}$

19. $5x - 10y - z = -20, \hat{\mathbf{n}} = (5\mathbf{i} - 10\mathbf{j} - \mathbf{k})/\sqrt{126}$

21. From Theorem 2.4(a) $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$. Make the substitutions $\mathbf{a}\to\mathbf{b}, \mathbf{b}\to\mathbf{c}$, and $\mathbf{c}\to\mathbf{a}$ to get $\mathbf{b}\times(\mathbf{c}\times\mathbf{a}) = (\mathbf{a}\cdot\mathbf{b})\mathbf{c} - (\mathbf{b}\cdot\mathbf{c})\mathbf{a}$. Now make the substitutions $\mathbf{b}\to\mathbf{c}$, $\mathbf{c}\to\mathbf{a}$, and $\mathbf{a}\to\mathbf{b}$ to get $\mathbf{c}\times(\mathbf{a}\times\mathbf{b}) = (\mathbf{b}\cdot\mathbf{c})\mathbf{a} - (\mathbf{a}\cdot\mathbf{c})\mathbf{b}$. The result follows by adding these results.

23. Yes **25.** Yes

27. Area of base $= 1/2$ area of parallelogram with sides $\mathbf{b}$ and $\mathbf{c}$, so $S = (1/2)\|\mathbf{b}\times\mathbf{c}\|$. Vertical height $h = \mathbf{a}\cdot\hat{\mathbf{n}}$, so volume of tetrahedron is $V = (1/3)hS = (1/6)|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|$.

29. Take the dot product with $\mathbf{b}\times\mathbf{c}$ to get $\lambda\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) + \mu\mathbf{b}\cdot(\mathbf{b}\times\mathbf{c}) + \nu\mathbf{c}\cdot(\mathbf{b}\times\mathbf{c}) + \mathbf{d}\cdot(\mathbf{b}\times\mathbf{c}) = 0$. The two middle terms are zero, so $\lambda\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) + \mathbf{d}\cdot(\mathbf{b}\times\mathbf{c}) = 0$. So, provided $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are linearly independent, $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) \ne 0$, so then $\lambda = -\mathbf{d}\cdot(\mathbf{b}\times\mathbf{c})/[\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})]$, and the other result follows in similar fashion.

31. Write Theorem 2.4 in the form $\mathbf{b}\times(\mathbf{c}\times\mathbf{d}) = (\mathbf{b}\cdot\mathbf{d})\mathbf{c} - (\mathbf{b}\cdot\mathbf{c})\mathbf{d}$ and form the dot product with $\mathbf{a}$ to obtain $\mathbf{a}\cdot[\mathbf{b}\times(\mathbf{c}\times\mathbf{d})] = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$. Interchanging the dot and cross on the left gives the result.

Exercise Set 2.5

1. Sum $(3, 0, 2, 4, 6)$, norms $\sqrt{13}, \sqrt{26}$, dot product 13

3. Sum $(0, 0, 0, 0, 0)$, norms $\sqrt{11}$, $\sqrt{11}$, dot product -11

5. Sum $(3, 2, 1, 4)$, norms $\sqrt{10}$, $\sqrt{20}$, dot product 0

7. Sum $(1, 1, -3, 0, 3)$, norms $\sqrt{22}$, $\sqrt{10}$, dot product -6

9. 0.859 rad, unit n-vectors $(1/\sqrt{15})(3, 1, 2, 1)$, $(1/\sqrt{10})(1, -1, 2, 2)$

11. 0 rad, unit n-vectors $(1/\sqrt{7})(1, -1, -1, 2)$, $(1/\sqrt{7})(1, -1, -1, 2)$

13. No. Null vector belongs to S but the summation and scaling laws fail.

15. No. The null vector is not contained in S and both the scaling and summation laws are not satisfied.

17. Yes.

19. Yes, since a linear equation and a constant are special cases of quadratic functions.

21. Yes.

23. No. As $f'(x) > 0$ the zero function does not belong to S, and the scaling law is not satisfied when $\lambda < 0$, for then $f'(x) < 0$.

25. The null vector $(0, 0, 0)$ in $\mathbf{R}^3$ does not belong to S.

27. $\|x + \lambda y\|^2 = (x + \lambda y) \cdot (x + \lambda y) = \|x\|^2 + 2\lambda(x \cdot y) + \lambda^2\|y\|^2$ and $\|x - \lambda y\|^2 = (x - \lambda y) \cdot (x - \lambda y) = \|x\|^2 - 2\lambda(x \cdot y) + \lambda^2\|y\|^2$. The result follows by addition of these equalities.

29. Corresponding components must be equal, or $\mathbf{x} = c\mathbf{y}$, $c > 0$.

Exercise Set 2.6

1. Linearly independent **9.** Linearly independent

3. Linearly independent **11.** Linearly dependent

5. Linearly independent **13.** Linearly independent

7. Linearly dependent **15.** Linearly dependent

17. $e_1 = (1, 1, 0, 0, 0)$, $e_2 = (1, 1, 1, 0, 0)$, $e_3 = (1, 1, 0, 1, 0)$, $e_4 = (1, 1, 0, 0, 1)$; dimension 4

19. $e_1 = (2, 2, 1, 0, 0)$, $e_2 = (2, 2, 1, 1, 0)$, $e_3 = (2, 2, 1, 0, 1)$; dimension 3

21. (a) $2 = 2(u + v)$ lies in V (b) No, because $\sin 2x = 2\sin x \cos x$ does not lie in V (c) $0 = 0u + 0v$ lies in V (d) $\cos 2x = \cos^2 x - \sin^2 x = u - v$ lies in V (e) $2 + 3x$ does not lie in V (f) Yes, because 3 and $-4\cos 2x$ both lie in V

Exercise Set 2.7

1. $\mathbf{i} + 2\mathbf{j} + \mathbf{k}$, $(7/6)\mathbf{i} - (2/3)\mathbf{j} + (1/6)\mathbf{k}$, $(5/11)\mathbf{i} + (5/11)\mathbf{j} - (15/11)\mathbf{k}$

3. $2\mathbf{i} + \mathbf{j}$, $-(4/5)\mathbf{i} + (8/5)\mathbf{j} + \mathbf{k}$, $(4/21)\mathbf{i} - (8/21)\mathbf{j} + (16/21)\mathbf{k}$

5. $-\mathbf{i} + \mathbf{k}$, $(1/2)\mathbf{i} + 2\mathbf{j} + (1/2)\mathbf{k}$, $(2/3)\mathbf{i} - (1/3)\mathbf{j} + (2/3)\mathbf{k}$

7. $\mathbf{a}_1 = 3\mathbf{j} - \mathbf{k}$, $\mathbf{a}_2 = \mathbf{i} + \mathbf{j}$, $\mathbf{a}_3 = \mathbf{i} + 2\mathbf{k}$. Starting with $\mathbf{u}_1 = \mathbf{a}_1$; $3\mathbf{j} - \mathbf{k}$, $\mathbf{i} + (1/10)\mathbf{j} + (3/10)\mathbf{k}$, $-(5/11)\mathbf{i} + (5/11)\mathbf{j} + (15/11)\mathbf{k}$. Rearrange with $\mathbf{a}_1 = \mathbf{i} + \mathbf{j}$, $\mathbf{a}_2 = 3\mathbf{j} - \mathbf{k}$, $\mathbf{a}_3 = \mathbf{i} + 2\mathbf{k}$; $\mathbf{i} + \mathbf{j}$, $-(3/2)\mathbf{i} + (3/2)\mathbf{j} - \mathbf{k}$, $-(5/11)\mathbf{i} + (5/11)\mathbf{j} + (15/11)\mathbf{k}$

Exercise Set 3.1

1. $a = -1, b = 3, c = 4$ **3.** $a = 1, b = 3, c = 2$

5. $\mathbf{A} + \mathbf{B} = \begin{bmatrix} 3 & 4 & 4 & 4 \\ 3 & 2 & -3 & 3 \\ 1 & 0 & 1 & 1 \end{bmatrix}$,

$\mathbf{A} - \mathbf{B} = \begin{bmatrix} -1 & 4 & 2 & 8 \\ 1 & 0 & 3 & 1 \\ 1 & -2 & -1 & 1 \end{bmatrix}$

7. $\mathbf{A} + \mathbf{B} = \begin{bmatrix} 1 & 4 & 7 \\ 6 & 0 & 1 \\ 1 & 2 & 1 \\ 3 & 5 & 6 \end{bmatrix}$, $\mathbf{A} - \mathbf{B} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

9. $\mathbf{A} + 3\mathbf{B} = \begin{bmatrix} 7 & 13 & -1 \\ 5 & 7 & 16 \\ 6 & 2 & 11 \end{bmatrix}$

11. $4\mathbf{A} - 2\mathbf{B} = \begin{bmatrix} 4 & 10 & 4 \\ 4 & -4 & 0 \\ 2 & 6 & 0 \end{bmatrix}$

13. 14 **15.** 15 **17.** $\mathbf{BA} = \begin{bmatrix} 6 & 17 & 7 \\ 4 & 2 & 6 \end{bmatrix}$

19. $\mathbf{AB} = \mathbf{BA} = \mathbf{B}$

21. $\mathbf{AB} = \begin{bmatrix} 17 & 8 & 25 \\ 24 & 16 & 30 \\ 20 & 28 & 56 \\ 17 & 10 & 37 \end{bmatrix}$

25. $\mathbf{A} = \begin{bmatrix} 4 & 5 & -1 & 7 \\ 3 & 2 & 0 & 3 \\ 0 & 1 & 6 & -7 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} u \\ v \\ w \\ z \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 25 \\ 6 \\ 0 \end{bmatrix}$

27. $\mathbf{A} = \begin{bmatrix} 3 - \lambda & 4 & -2 \\ 2 & -7 - \lambda & 6 \\ 8 & 3 & 5 - \lambda \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $\mathbf{b} = \mathbf{0}$

29. $\mathbf{X} = \frac{1}{4}\begin{bmatrix} 11 & -1 & 1 \\ 25 & 7 & 19 \\ 12 & -7 & 9 \end{bmatrix}$

43. Use $\mathbf{A}^4 = \mathbf{A}^2\mathbf{A}^2$ and $\mathbf{A}^6 = \mathbf{A}^2\mathbf{A}^4$ to show that $\mathbf{A}^6 = \mathbf{I}$.

45. $(p = 0, q = 0, r = 1), (p = 0, q = 1, r = 0), (p = 1, q = 0, r = 0)$

46. $n = 3$

47. The structure of $\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x}$ is such that it is a sum of products of the form $x_i x_j$ with $i, j = 1, 2, 3$. $\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x} = 3x_1^2 + 8x_1x_2 + 6x_1x_4 + 2x_2^2 + 4x_2x_3 + 12x_2x_4 + 5x_3^2 + 2x_3x_4 + 7x_4^2$.

51. $\begin{bmatrix} 2 & 2 & 7/2 \\ 2 & 6 & 0 \\ 7/2 & 0 & -9 \end{bmatrix}$

53. Use the fact that $\mathbf{PE} = \mathbf{E}$, so $\mathbf{P}^2\mathbf{E} = \mathbf{PE} = \mathbf{E}$, etc.

Exercise Set 3.2

1. (a) Yes (b) No, there is one negative entry (c) No, second row sum >1 (d) Yes

3. $\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}$ **5.** $\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$

Exercise Set 3.3

1. $\det\mathbf{A} = -7$ **3.** $\det\mathbf{A} = 43$

13. $x_1 = -7, x_2 = -11, x_3 = -15$

15. $P(\lambda) = -\lambda^3 + 6\lambda^2 - 3\lambda - 10$; $P(\lambda) = 0$ when $\lambda = -1, 2, 5$

17. $\det\mathbf{A} = -14, \det\mathbf{B} = -18, \det(\mathbf{AB}) = 252$

Exercise Set 3.5

5. $\mathbf{E}_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\mathbf{E}_2(3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$,

$\mathbf{E}_{12}(6) = \begin{bmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

7. $\begin{bmatrix} 1 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1/4 \end{bmatrix}$ **9.** $\begin{bmatrix} 1 & 0 & 0 & 3/2 & 1 \\ 0 & 1 & 0 & 2/3 & 0 \\ 0 & 0 & 1 & -5/6 & -3/2 \end{bmatrix}$

11. $\begin{bmatrix} 1 & 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix}$

13. $\begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 8 \end{bmatrix}$, $x_1 = -4, x_2 = 0, x_3 = 8$

15. $\begin{bmatrix} 1 & 0 & 0 & -1 & -2 \\ 0 & 1 & 0 & 2 & 2 \\ 0 & 0 & 1 & -3 & -3 \end{bmatrix}$, $x_1 = k - 2, x_2 = 2 - 2k, x_3 = -3 + 3k, x_4 = k$

17. $\begin{bmatrix} 1 & 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$, no solution

Exercise Set 3.6

1. $\begin{bmatrix} 1 & 0 & 0 & -2 & -1/2 & -1 \\ 0 & 1 & 0 & -2 & 1/2 & -1 \\ 0 & 0 & 1 & 8 & 0 & 5 \end{bmatrix}$,

rank = 3, row space {[1, 0, 0, −2, −1/2, −1], [0, 1, 0, −2, 1/2, −1], [0, 0, 1, 8, 0, 5]}
column space $\{[0, 0, 1]^{\mathrm{T}}, [1, 0, 0]^{\mathrm{T}}, [0, 1, 0]^{\mathrm{T}}\}$

3. $\begin{bmatrix} 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

rank = 4, row space {[0, 0, 0, 0, 1], [1, 0, 0, 2, 0], [0, 0, 1, 0, 0], [0, 1, 0, 3, 0]}
column space $\{[0, 0, 0, 1]^{\mathrm{T}}, [0, 0, 1, 0]^{\mathrm{T}}, [0, 1, 0, 0]^{\mathrm{T}}, [1, 0, 0, 0]^{\mathrm{T}}\}$

5. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

rank = 3, row space {[1, 0, 0], [0, 1, 0], [0, 0, 1]}
column space $\{[1, 0, 0]^{\mathrm{T}}, [0, 1, 0]^{\mathrm{T}}, [0, 0, 1]^{\mathrm{T}}\}$

7. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

rank = 3, row space {[1, 0, 0], [0, 1, 0], [0, 0, 1]},
column space $\{[0, 1, 0, -2/13]^{\mathrm{T}}, [0, 0, 1, -7/13]^{\mathrm{T}}, [1, 0, 0, 20/13]^{\mathrm{T}}\}$

9. $\begin{bmatrix} 1 & 0 & -1/3 & -2/3 & -1/3 & -5/3 \\ 0 & 1 & 2/3 & 7/3 & 8/3 & 13/3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

rank = 2, row space {[0, 1, 2/3, 7/3, 8/3, 13/3],

$[1, 0, -1/3, -2/3, -1/3, -5/3]\}$,
column space $\{[0, 1, 1]^T, [1, 0, 1]^T\}$

11. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

rank $= 4$, row space $\{[1, 0, 0, 0], [0, 1, 0, 0], [0, 0, 1, 0], [0, 0, 0, 1]\}$,
column space $\{[1, 0, 0, 0]^T, [0, 1, 0, 0]^T, [0, 0, 1, 0]^T, [0, 0, 0, 1]^T\}$

13. $\begin{bmatrix} 1 & 7 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

rank $= 3$, row space $\{[0, 0, 1, 0], [0, 0, 0, 1], [1, 7, 0, 0]\}$,
column space $\{[0, 1, 0]^T, [1, 0, 0]^T, [0, 0, 1]^T\}$

Exercise Set 3.7

1. $x_1 = -2a - 6b, \quad x_2 = a + 4b, \quad x_3 = -a - (7/2)b, \quad x_4 = a, x_5 = b$

3. $x_1 = k, x_2 = -k, x_3 = 0, x_4 = k$

5. $x_1 = x_2 = x_3 = 0$

7. $x_1 = -(1/4)a + (5/4)b - (3/4)c$,
$x_2 = (1/20)a - (29/20)b + (7/20)c$,
$x_3 = (3/20)a - (7/20)b + (1/20)c$,
$x_4 = -(13/20)a + (37/20)b - (31/20)c$,
$x_5 = a, x_7 = b, x_7 = c$

9. $x_1 = (4/9)a + (37/9)b - (14/9)c, \quad x_2 = -a - 3b, \quad x_3 = (1/9)a - (2/9)b + (1/9)c, \quad x_4 = a, \quad x_5 = b, \quad x_6 = c$

Exercise Set 3.8

1. $x_1 = 3, x_2 = 1, x_3 = -2, x_4 = 4$

3. $x_1 = -5/12, x_2 = -1/12, x_3 = 1/6, x_4 = 1/2$

5. Inconsistent; no solution

7. $x_1 = -15/11, x_2 = 1/11, x_3 = 8/11, x_4 = 5/11$

9. Inconsistent: no solution

Exercise Set 3.9

1. $\begin{bmatrix} -1/5 & 4/15 & 1/3 \\ 2/5 & 3/10 & -1/2 \\ 0 & -1/6 & 1/6 \end{bmatrix}$

3. $\begin{bmatrix} -2/73 & 16/73 & -5/73 \\ -9/73 & -1/73 & 14/73 \\ 28/73 & -5/73 & -3/73 \end{bmatrix}$

5. $\begin{bmatrix} 2 & 1 & -2 \\ -1 & 0 & 1 \\ 0 & -2 & 1 \end{bmatrix}$

7. $\begin{bmatrix} 37/131 & 8/131 & -31/131 & 43/131 \\ 52/131 & -10/131 & 6/131 & -21/131 \\ -1/131 & -25/131 & 15/131 & 13/131 \\ -10/131 & 12/131 & 19/131 & -1/131 \end{bmatrix}$

9. $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$
$$= \begin{bmatrix} 31/276 & 1/207 & -7/69 \\ -19/276 & 1/414 & -7/138 \\ -4/69 & 19/414 & 5/138 \end{bmatrix}$$

11. $\begin{bmatrix} 25/27 & -31/27 & 13/9 \\ -7/27 & 13/27 & -4/9 \\ -1/27 & -2/27 & 2/9 \end{bmatrix}$

13. $\begin{bmatrix} -2/27 & -1/9 & 16/27 \\ 28/27 & 5/9 & -89/27 \\ -11/27 & -1/9 & 34/27 \end{bmatrix}$

15. $\begin{bmatrix} 27/29 & -7/29 & -1/58 & -7/58 \\ -28/29 & 18/29 & 15/58 & -11/58 \\ -3/29 & 4/29 & -8/29 & 2/29 \\ -11/29 & 5/29 & 9/58 & 5/58 \end{bmatrix}$

17. Elementary row operations require far less computational effort.

Exercise Set 3.10

1. $\dfrac{d\mathbf{C}}{dt} = \begin{bmatrix} 3t^2 & 1+4t & \sin t + t\cos t + \sinh t \\ 1+2t & -\sin t & 2\cos 2t - \sin t \end{bmatrix}$

$\dfrac{d^2\mathbf{C}}{dt} = \begin{bmatrix} 6t & 4 & 2\cos t - t\sin t + \cosh t \\ 2 & -\cos t & -4\sin 2t - \cos t \end{bmatrix}$

3. $\dfrac{d\mathbf{C}}{dt} = \begin{bmatrix} 1-4e^{2t} & 0 & -3t^2 \\ 0 & 3-4t & 2e^{2t} - 2\cosh t \end{bmatrix}$

$\dfrac{d^2\mathbf{C}}{dt^2} = \begin{bmatrix} -8e^{2t} & 0 & -6t \\ 0 & -4 & 4e^{2t} - 2\sinh t \end{bmatrix}$

7. $\dfrac{d\mathbf{A}^{-1}}{dt} =$
$$\begin{bmatrix} -\sin t & -\cos t & 0 \\ \cos t & -\sin t & 0 \\ -\sin t - 3t\cos t + t^2\sin t & -\cos t + 3t\sin t + t^2\cos t & 0 \end{bmatrix}$$

9. As $\mathbf{AA}^{-1} = \mathbf{I}$, $\dfrac{d}{dt}(\mathbf{AA}^{-1}) = \dfrac{d\mathbf{A}}{dt}\mathbf{A}^{-1} + \mathbf{A}\dfrac{d\mathbf{A}^{-1}}{dt} = \mathbf{0}$, so another differentiation gives $\frac{d^2\mathbf{A}}{dt^2}\mathbf{A}^{-1} + 2\frac{d\mathbf{A}}{dt}\frac{d\mathbf{A}^{-1}}{dt} + \mathbf{A}\frac{d^2\mathbf{A}^{-1}}{dt^2} = \mathbf{0}$. Now substitute for $\frac{d\mathbf{A}^{-1}}{dt}$ to find $\frac{d^2\mathbf{A}^{-1}}{dt^2}$.

Exercise Set 4.1

1. $P(\lambda) = \lambda^3 - 3\lambda^2$
3. $P(\lambda) = \lambda^3 - 3\lambda^2 + 5\lambda + 1$
5. $P(\lambda) = \lambda^3 - 4\lambda^2 - 2\lambda$
7. $P(\lambda) = \lambda(\lambda - 1)(\lambda^2 - \lambda - 2)$
9. $1, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix};\ 2, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix};\ -1, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$
11. $-1, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix};\ 1, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix};\ 3, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$
13. $-2, \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix};\ 1, \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix};\ 0, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$
15. $1, \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix};\ 2, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix};\ -2, \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$
17. $1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix};\ 2, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix};\ 0, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$
19. $2, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix};\ 1, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix};\ 1, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$
21. $0, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix};\ 2, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix};\ 2, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$
23. $P(\lambda) = (\lambda + 1)(\lambda^3 - \lambda^2 - 4\lambda + 4)$;
$1, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix};\ 2, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix};\ -2, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix};\ -1, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$
25. To obtain the first result expand the characteristic determinant in terms of elements of the first column. The second part of the problem is illustrated by $\mathbf{A} = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$ with eigenvalues $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = 2$ and eigenvectors $\mathbf{x}_{1,2} = [1,\ 0,\ 0]^{\mathrm{T}}$ and $\mathbf{x}_3 = [-3,\ 2,\ 1]^{\mathrm{T}}$.
31. Premultiplication of a matrix by $\mathbf{E}$ interchanges its ith and jth rows, while premultiplication by $\mathbf{E}^{\mathrm{T}}$ reverses the process. Thus as $\mathbf{E}$ is obtained from $\mathbf{I}$, it follows that $\mathbf{E}^{\mathrm{T}}\mathbf{E} = \mathbf{I}$. This shows that $\mathbf{E}^{\mathrm{T}} = \mathbf{E}^{-1}$, and so $\mathbf{E}$ is an orthogonal matrix. As the product of two orthogonal matrices is an orthogonal matrix, if $\mathbf{Q}$ is an orthogonal matrix, so also is the matrix $\mathbf{EQ}$ obtained from $\mathbf{Q}$ by a row interchange. Multiplication of $\mathbf{Q}$ by a sequence of elementary matrices $\mathbf{E}_1, \mathbf{E}_2, \ldots, \mathbf{E}_t$ will interchange the rows of $\mathbf{Q}$ in any desired order while leaving the result still an orthogonal matrix.

Exercise Set 4.2

In solutions 1 through 12 a diagonalizing matrix $\mathbf{P}$ is formed by using the given eigenvectors in any order as the columns of $\mathbf{P}$. The elements on the leading diagonal of the corresponding diagonal matrix are then arranged in the same order as the eigenvectors to which they belong.

1. $1, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix};\ -1, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix};\ 2, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$
3. $2, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix};\ -1, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix};\ 1, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$
5. $1, \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix};\ 1, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix};\ -1, \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$
7. $1, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix};\ 3, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix};\ 3, \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$
9. $1, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix};\ 2, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix};\ -1, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$
11. $0, \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix};\ -2, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix};\ -2, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$
13. $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1/3 \\ -1/3 \\ 2/3 \end{bmatrix}, \begin{bmatrix} -1/2 \\ 1/2 \\ 0 \end{bmatrix}$
15. $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3/\sqrt{18} \\ 3/\sqrt{18} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$
17. $3, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix};\ 2, \begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix};\ 4, \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

19. Two equal eigenvalues, but the corresponding eigenvectors are orthogonal:
$$5, \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}; \ 3, \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}; \ 3, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

21. Two equal eigenvalues, but the corresponding eigenvectors are orthogonal:
$$6, \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}; \ 2, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}; \ 2, \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}$$

25. $P(\lambda) = \lambda^2 - 6\lambda + 11, \quad \mathbf{A}^{-1} = \begin{bmatrix} 4/11 & -3/11 \\ 1/11 & 2/11 \end{bmatrix}$

27. $P(\lambda) = -\lambda^3 + \lambda^2 - 4,$
$$\mathbf{A}^{-1} = \begin{bmatrix} 0 & -1/2 & -1/2 \\ 1 & 1 & 1 \\ -1/2 & -1/2 & -1 \end{bmatrix}$$

Exercise Set 4.3

1. $\begin{bmatrix} 1 & 1-i & 2i \\ 1+i & 2 & 3-i \\ -2i & 3+i & 4 \end{bmatrix} + \begin{bmatrix} i & 2+2i & 3 \\ -2+2i & 0 & 1+2i \\ -3 & -1+2i & 2i \end{bmatrix}$

Hermitian skew-Hermitian

3. $\begin{bmatrix} 4 & 2i & 1+i \\ -2i & 1 & 3 \\ 1-i & 3 & 0 \end{bmatrix} + \begin{bmatrix} -2i & 1-i & 1+i \\ -1-i & 2i & 1 \\ -1+i & -1 & 0 \end{bmatrix}$

Hermitian skew-Hermitian

5. $\frac{3}{2} \pm \frac{\sqrt{21}}{2}$ **9.** $\frac{1}{2}i(3 \pm \sqrt{41})$

7. $2 \pm \sqrt{14}$ **11.** $\pm i\sqrt{13}$

13. $(1+i)/\sqrt{2}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}; \ (1-i)/\sqrt{2}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

15. $i, \begin{bmatrix} i \\ 1 \end{bmatrix}; \ 1, \begin{bmatrix} -i \\ 1 \end{bmatrix}$

Exercise Set 4.4

1. $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & -3 \\ 2 & -3 & -2 \end{bmatrix}$ **3.** $\begin{bmatrix} -2 & 0 & -1 \\ 0 & 3 & 2 \\ -1 & 2 & 0 \end{bmatrix}$

5. $\begin{bmatrix} 3 & -2 & 0 & 0 \\ -2 & 0 & -3 & -1 \\ -3 & -3 & 2 & 0 \\ 0 & -1 & 0 & 8 \end{bmatrix}$

7. $2x_1^2 + x_2^2 - x_3^2 + 3x_4^2 + 8x_1x_2 + 8x_1x_3 + 4x_2x_3 + 2x_2x_4 + 4x_3x_4$

9. $3x_2^2 + 2x_3^2 + 7x_4^2 + 4x_1x_2 - 8x_1x_3 + 4x_1x_4 + 2x_2x_3 + 2x_3x_4$

11. $\mathbf{Q} = \begin{bmatrix} 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix},$

$\mathbf{P} = y_1^2 + 3y_2^2 + y_3^2$, $\mathbf{y} = \mathbf{Q}^{\mathrm{T}}\mathbf{x}$, positive definite

13. $\mathbf{Q} = \begin{bmatrix} 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix},$

$\mathbf{P} = 3y_1^2 + 4y_2^2 + 5y_3^2$, $\mathbf{y} = \mathbf{Q}^{\mathrm{T}}\mathbf{x}$, positive definite

15. $\mathbf{Q} = \begin{bmatrix} 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix},$

$\mathbf{P} = -y_1^2 + y_2^2 + 2y_3^2$, $\mathbf{y} = \mathbf{Q}^{\mathrm{T}}\mathbf{x}$, indefinite

17. $\mathbf{Q} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix},$

$\mathbf{P} = 2y_1^2 + 3y_2^2 - y_3^2$, $\mathbf{y} = \mathbf{Q}^{\mathrm{T}}\mathbf{x}$, indefinite

19. Ellipse **21.** Hyperbolic **23.** Ellipse

25. $\mathbf{A} = \begin{bmatrix} 1 & 4 \\ 4 & 1 \end{bmatrix}, \quad \mathbf{Q} = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix},$

$\mathbf{D} = \begin{bmatrix} -3 & 0 \\ 0 & 5 \end{bmatrix}, \quad \mathbf{x} = \mathbf{Q}\mathbf{y}, \quad x_1 = (-y_1 + y_2)/\sqrt{2},$
$x_2 = (y_1 + y_2)/\sqrt{2}, -3y_1^2 + 5y_2^2$

27. $\mathbf{A} = \begin{bmatrix} -2 & 2 \\ 2 & 1 \end{bmatrix}, \quad \mathbf{Q} = \begin{bmatrix} -2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix},$

$\mathbf{D} = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}, \quad \mathbf{x} = \mathbf{Q}\mathbf{y}, \quad x_1 = -(2/\sqrt{5})y_1 + (1/\sqrt{5})y_2, \quad x_2 = (1/\sqrt{5})y_1 + (2/\sqrt{5})y_2, -3y_1^2 + 2y_2^2 - (9/\sqrt{5})y_1 + (2/\sqrt{5})y_2$

29. $\mathbf{A} = \begin{bmatrix} 35/17 & 4/17 \\ 4/17 & 50/17 \end{bmatrix}, \quad \mathbf{Q} = \begin{bmatrix} -4/17 & 1/\sqrt{17} \\ 1/\sqrt{17} & 4/\sqrt{17} \end{bmatrix},$

$\mathbf{D} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \quad \mathbf{x} = \mathbf{Q}\mathbf{y}, \ x_1 = (-4y_1 + y_2)/\sqrt{17}$
$x_2 = (y_1 + 4y_2)/\sqrt{17}, \quad 2y_1^2 + 3y_2^2 + (4/\sqrt{17})y_1 + (16/\sqrt{17})y_2$

Exercise Set 4.5

1. $n = 4$ **5.** $e^{\mathbf{A}t} = \begin{bmatrix} e^{mt} & 0 \\ 0 & e^{nt} \end{bmatrix}$

7. $e^{\mathbf{A}t} = \begin{bmatrix} \frac{4}{5}e^{-3t} + \frac{1}{5}e^{2t} & -\frac{2}{5}e^{-3t} + \frac{2}{5}e^{2t} \\ -\frac{2}{5}e^{-3t} + \frac{2}{5}e^{2t} & \frac{1}{5}e^{-3t} + \frac{4}{5}e^{2t} \end{bmatrix}$

9. $e^{At} = \begin{bmatrix} 2e^t - e^{2t} & -e^t - e^{2t} & 2e^t - 2e^{2t} \\ 2e^t - 2 & 2 - e^t & 2e^t - 2 \\ e^{2t} - 1 & 1 - e^{2t} & 2e^{2t} - 1 \end{bmatrix}$

11. Follows from the definitions.

Exercise Set 5.1

1. Homogeneous linear of order 3 and degree 1
3. Nonlinear of order 2 and degree 1
5. Nonlinear of order 2 and degree 1
7. Nonhomogeneous linear of order 1 and degree 1
9. Nonlinear of order 1

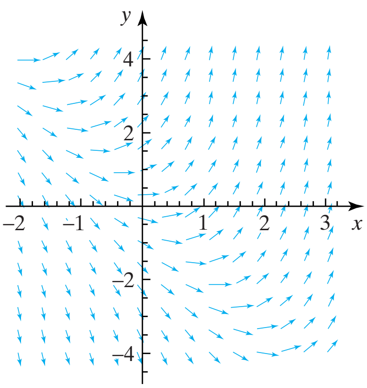
Exercise Set 5.2

1. $\left(y - x\frac{dy}{dx}\right)^2 = 2xy\left(1 + \left(\frac{dy}{dx}\right)^2\right)$

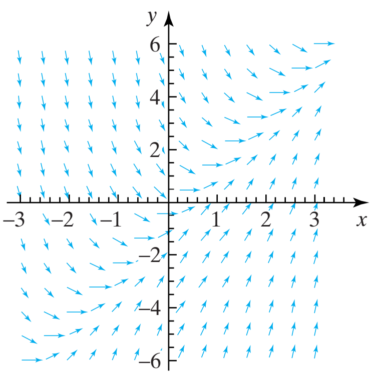
3. $x\frac{d^2y}{dx^2} = \frac{U}{V}\left(1 - \left(\frac{dy}{dx}\right)^2\right)^{1/2}$

Exercise Set 5.3

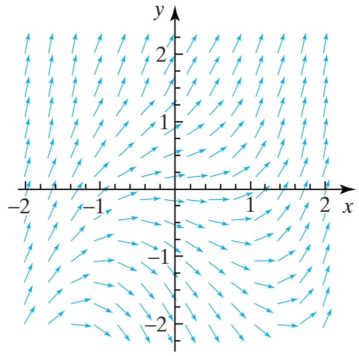
1.



3.



5.



Exercise Set 5.4

1. $x^2 + 2y + \ln|2y - 1| = 3$; Singular solution $y = 1/2$ does not satisfy $y(1) = 1$
3. $y = (x^2 - 3)/[2(x^2 - 4)]$
5. $\ln|y + \sqrt{(y^2 - 1)}| = 3(1 + x^2)^{1/2} + C$
7. $2\ln|y + 2| + 2/(y + 2) = C - \ln|x + 1|$
9. $2\ln|y| + 3y^2 = 4x - 4(x + 1)\ln|x + 1| + C$
11. $\ln[(1 + x^2)/(y^2 + y + 1)] + (2/\sqrt{3})\text{Arctan}[(2y + 1)/\sqrt{3}] = C$
13. $y = 2 + C\cos^2 x$
15. Eliminate k between the original equation and $dy/dx = -1/k$ to obtain the differential equation of the orthogonal trajectories $dy/dx = -(x - a)/(y - b)$, with the solution $x^2 + y^2 - 2ax - 2by = C$, the equation of a family of concentric circles with their center at (a, b).
17. Eliminate C between the original equation and $dy/dx = -1/\{2Cxe^{2x}(1 + x)\}$ to obtain the differential equation of the orthogonal trajectories $dy/dx = -x/\{2y(1 + x)\}$, with the solution $y^2 = -x + \ln|1 + x| + C$.
19. $\lambda = \ln(N_2/N_1)/(t_2 - t_1)$; predicts infinite growth
21. Approximately 50,200 years

Exercise Set 5.5

1. $x/y^2 + 1/y = C$ **3.** $y = x(4\ln|x| + C)^{1/2}$
5. $-(1/2)x^2 + xy + y^2 = C$
7. $-2\ln|x| + (1/2)\cos(y/x)\sin(y/x) + (1/2)y/x = C$
9. $-\ln|x| - (1/2)\cos(y/x)\sin(y/x) + (1/2)y/x = C$
11. $x/(y + 2) - \ln|y + 2| = C$
13. $x + 1 = [C(1 + x)\exp\{\text{Arctan}[y/(1 + x)]\}]/[y^2 + (1 + x)^2]^{1/2}$

Exercise Set 5.6

1. (a) Not exact
(b) $f(x, y) = x^4 + \sin x + 3xy^2 + 2y = C$

3. (a) $f(x, y) = x \sin x + y^3 + \sinh(x + 2y) = C$
(b) Not exact

5. (a) $f(x, y) = (x^3 + y^2)^{1/2} + 3y^2 = C$
(b) $f(x, y) = y \ln x + x^2 \sinh(y^2) = C$

7. (a) $x^2 y + 6 \ln x + 4 \ln y = C$
(b) $f(x, y) = x^2/(2x + 3y^2) + 2x = C$

Exercise Set 5.7

1. $y = 1/2 + Ce^{-2x}$

3. $y = (1/3)(2x^3 + 3x^2 + 3C)/(x + 1)$

5. $y = (1/6)(6Cx^3 - 3x - 2)/x$

7. $y = (1/4)(4C + x^4)/x^2$

9. $y = \sin x\{C + 2\ln(\cos x - 1)\}/(1 + \cos x)$

11. $y = x \sin x + x$ **13.** $y = 2x^2 - x - 1$

14. $y = x^4/3 + 2/(3x^2)$

15. $y = x/\sin x - \pi/(2 \sin x) - \cos x$

17. Approximately 173 seconds

19. $dv/dt + kv + kt = 0$; $v(t) = \frac{(v_0 k - 1)}{k} e^{-kt} + \frac{1}{k} - t$; $k = \frac{4}{(4 - e)v_0}$

Exercise Set 5.8

1. $y^{1/2} = x - 1 + Ce^{-x}$

3. $y^{1/2} = 1/(4 - 2x + Ce^{-x/2})$

5. $y = 1/(1 + Ce^{-2\cos x})$

7. $y^{1/2} = 4x/(4C - x^2)$

9. $n(t) = \frac{n_0 a}{n_0 b + (a - n_0 b)e^{-at}}$. If $a/b = n_0$, then $n(t) = n_0$ (constant); otherwise $n(t)$ approaches the value a/b. Thus, if $a/b > n_0$ the stock level increases to a value greater than n_0, and if $a/b < n_0$ it decreases to a value less than n_0.

Exercise Set 5.9

3. $y = x + \exp(2x^2/3)/\{C - 2\int x \exp(2x^3/3)dx\}$

5. $y = 1 + 1/(Ce^{-x} - 2)$

Exercise Set 5.10

1. Initial conditions can be imposed anywhere in the part of the plane $x < 1$ other than on the line $x = 1$, where $\partial f/\partial x$ is infinite.

3. Initial conditions can be imposed anywhere in the (x, y)-plane.

5. Initial conditions can be imposed anywhere in the (x, y)-plane other than on the y-axis.

Exercise Set 6.1

1. (a) Linearly independent (b) Linearly independent (c) Linearly independent

3. (a) Linearly independent (b) Linearly independent (c) Linearly dependent

5. $y = c_1 e^x + c_2 e^{-4x}$ **9.** $y = c_1 e^x + c_2 e^{-3x}$

7. $y = e^x(c_1 \cos x + c_2 \sin x)$ **11.** $y = (c_1 + c_2 x)e^{-3x}$

13. $y = e^{2x}(c_1 \cos x + c_2 \sin x)$

15. $y = e^{-3x}(c_1 \cos 4x + c_2 \sin 4x)$

17. $y = c_1 e^{-4x} + c_2 e^{-x}$

19. $y = e^{3x/2}\{c_1 \cos(x\sqrt{3}/2) + c_2 \sin(x\sqrt{3}/2)\}$

21. $y = 5e^{-2x} - 4e^{-3x}$

23. $y = e^{-x}(3 \cos x + 4 \sin x)$ **25.** $y = 5e^{2x} - 3e^{3x}$

27. $y = e^{4x}/5 - 6e^{-x}/5$

29. $y = 3e^{-x}/(3 - e^2) - e^{-3x}/(3e^{-2} - 1)$

31. $y = (1/5)e^{-3(1+x)}(2 - 3x)$

33. $y = e^{-x}\{\cos 5x + (3/2) \sin 5x\}$

35. $y = e^{-2x}/(3e^{-3} - 2e^{-2}) - e^{-3x}/(3e^{-3} - 2e^{-2})$

37. (a) Not unique (b) No solution (c) Unique

39. $y = b \sin \lambda x$, b arbitrary and $\lambda = 0, \pm 1, \pm 2, \ldots$.

41. $\theta(t) = (\alpha/p)\exp(-kt) \sin pt$, and so the angular velocity is $d\theta/dt = -(ak/p)\exp(-kt)\sin(pt) + a\exp(-kt)\cos pt$. The pendulum comes to rest for the first time when $d\theta/dt$ first becomes zero. This occurs at the smallest positive value $t = t_C$, say, such that $\tan pt_C = p/k$. The angular displacement at $t = t_C$ is given by $\theta(t_C) = a\exp(-kt_C)/(k^2 + p^2)^{1/2}$.

Exercise Set 6.2

1. $y_p = (2/5) \sin x - (1/5) \cos x$,
$y_c = (2/5)(3 \cos 2x + \sin 2x)e^{-x}$

3. $y_p = -(1/2) \cos x$, $y_c = (3/2)(1 + x)e^{-x}$

5. $y_p = -(1/130)(9 \cos 3x + 7 \sin 3x)$,
$y_c = (13/10)e^{-x} - (16/13)e^{-2x}$

7. $y_p = (A/10)(\sin x - \cos x)$,
$y_c = (A/5 + 10)e^{-2x} - (A/10 + 7)e^{-3x}$

9. $y_p = (1/5)(\cos x + \sin x)$,
$y_c = (4/5)(4e^{-2x} - 3e^{-3x})$, $\tan \phi = -1$

11. $y_p = (1/9) \sin 3x$,
$y_c = \{2 + (23/3)x\}e^{-3x}$, $\phi = 0$

13. $y_p = (3/40)(\cos 2x + 3\sin 2x)$,
$y_c = (65/8)e^{-2x} - (21/5)e^{-4x}$, $\tan\phi = -1/3$

15. $$y(t) = 2000 - \frac{m}{10} - \frac{mt}{10} + \frac{m^2}{3200} + \left(\frac{m}{10} - \frac{m^2}{3200}\right)\exp\left(-\frac{320t}{m}\right),$$
$$\frac{dy}{dt} = -\frac{m}{10} - \frac{320}{m}\left(\frac{m}{10} - \frac{m^2}{3200}\right)\exp\left(-\frac{320t}{m}\right).$$
After a long fall the terminal speed is $|dy/dt| = m/10$, so setting $|dy/dt| = 24$ shows that $M = 240$ lbs.

17. $$x(t) = \frac{2}{9}\frac{g(\rho_2 - \rho_1)}{\eta}a^2 t + \frac{4}{81}\frac{g(\rho_2 - \rho)a^4\rho_1}{\eta^2} \times \left[\exp\left(-\frac{9}{2}\frac{\eta t}{a^2\rho_1}\right) - 1\right]$$
The container reaches the surface at a time $t = T$ given by $x(T) = h$. As it will reach its terminal speed soon after release, the exponential term can be ignored so $T \approx 9\eta h/[2g(\rho_2 - \rho_1)a^2]$.

19. Try, for example, $\omega_1 = 1$ and $\omega_2 = 1.05$ with $0 \le t \le 20$. Use the result $\cos\omega_1 t + \cos\omega_2 t = 2\cos\{\frac{1}{2}(\omega_1 + \omega_2)t\}\cos\{\frac{1}{2}(\omega_1 - \omega_2)t\}$. The high frequency component is the term with argument $\frac{1}{2}(\omega_1 + \omega_2)t$, and this is modulated by the term with argument $\frac{1}{2}(\omega_1 - \omega_2)t$.

Exercise Set 6.3

3. Not linearly independent; $(1 + 2x)^2$ is a linear combination of 3, $-x$ and x^2

5. $y = c_1\cosh x^2 + c_2\sinh x^2$ (for all x)

7. General solution: $y = c_1 e^x + (c_2\cos 3x + c_3\sin 3x)e^{-2x}$ (for all x); solution of i. v. p. is $y = (13/18)e^x + (5/18)e^{-2x}\cos 3x - (1/18)e^{-2x}\sin 3x$

9. $y = c_1 x + c_2(8x^2 - 1)$ (for all x)

11. $y = c_1 x + c_2\sin(x/2)$ (for all x)

13. $3/4 + (1/68)[9\sqrt{17}\sinh(x\sqrt{17}/2) + 17\cosh(x\sqrt{17}/2)]e^{-x/2}$

15. $y = ((5/4) + (1/2)\sin 2x - (1/4)\cos 2x)e^{-x}$

17. $y = (1/3)\cosh(x\sqrt{2}) + (2/3)\cos x$

19. $x(t) = A\cos(\omega_1 t - \phi) + B\cos(\omega_2 t - \psi)$, $y(t) = A\sin(\omega_1 t - \phi) - B\sin(\omega_2 t - \psi)$, with $\omega_1 = \frac{1}{2}(\sqrt{4c^2 + a} + a)$, $\omega_2 = \frac{1}{2}(\sqrt{4c^2 + a} - a)$. If the initial conditions make $B = 0$, the motion is in a circle with angular speed ω_1, whereas if they make $A = 0$ the motion is also in a circle, but this time in the opposite sense with angular speed ω_2.

Exercise Set 6.4

1. $y = -(14/9) - (1/3)x + (4/5)e^{2x} + c_1 e^x + c_2 e^{-3x}$

3. $y = 5 + (3/8)e^x - (1/2)xe^x + (1/4)x^2 e^x + c_1 e^{-x} + c_2 x e^{-x}$

5. $y = -(2/5)\cos x - (1/5)\sin x + c_1 e^{-2x} + c_2 x e^{-2x}$

7. $y = (1/2)x + e^{-x} + c_1 e^{-x}\cos x + c_2 e^{-x}\sin x$

9. $y = -x + 2x^2 - (2/3)x^3 + c_1 + c_2 e^{-x}\cos x + c_3 e^{-x}\sin x$

11. $y = (7/144) + (1/12)x + (1/2)e^{2x} - e^{3x} - xe^{3x} + c_1 e^{3x} + c_2 e^{4x}$

13. $y = -(9/80)x\cos 4x + (3/80)x\sin 4x + (57/1600)\sin 4x + (3/200)\cos 4x + c_1 e^{-4x} + c_2 e^{2x}$

15. $y = (1/18)\cos 3x + (1/36)\sin 3x - (1/6)x\cos 3x + (1/3)x\sin 3x + c_1\cos 3x + c_2\sin 3x$

17. $y = (7/4) - (3/2)x + (1/2)x^2 - 3e^{-2x} - 3xe^{-2x} + c_1 e^{-x} + c_2 e^{-2x}$

19. $y = -(1/2)xe^{-2x}\cos x + c_1 e^{-2x}\cos x + c_2 e^{-2x}\sin x$

21. $y = (1/3)e^{-3x}\cos x + (5/3)e^{-3x}\cos 2x + (7/2)e^{-3x}\sin 2x$

23. $y = (7/9) - (16/9)\cos 3x + (4/9)\sin 3x - (1/3)x\cos 3x - (2/3)x\sin 3x$

25. $y = (1/5) + (1/8)e^{-x} + (67/40)e^x\cos 2x - (11/40)e^x\sin 2x$

27. $y = -(3/2) - (3/5)\cos x - (1/5)\sin x + e^x + (11/10)e^{-x}\cos x + (13/10)e^{-x}\sin x$

Exercise Set 6.5

1. $y = c_1 x + c_2/x^3$

3. $y = (c_1/x^2)\cos(\sqrt{5}\ln|x|) + (c_2/x^2)\sin(\sqrt{5}\ln|x|)$

5. $y = c_1 x^2 + c_2/x^4$

7. $y = c_1/x + c_2/x^4$

9. $y = (c_1/x^{2/3})\cos(\frac{1}{2}\sqrt{7}\ln|x|) + (c_2/x^{2/3})\sin(\frac{1}{2}\sqrt{7}\ln|x|)$

11. The general solution is given in Solution 3.

13. $y = c_1 x + c_2 x^2 + c_3 x^3$

Exercise Set 6.6

1. $y = c_1 e^x + c_2 e^{-2x} + (1/27)e^x - (1/9)xe^x + (1/6)x^2 e^x$

3. $y = c_1 e^{-2x} + c_2 e^{-3x} - 2e^{-2x} + 2xe^{-2x} - x^2 e^{-2x} + (1/3)x^3 e^{-2x}$

5. $y = c_1e^x + c_2xe^x - 2xe^x + 2xe^x \ln|x|$

7. $y = c_1e^{-2x}\cos x + c_2e^{-2x}\sin x + (1/4)xe^{-2x}\cos x + (1/4)x^2e^{-2x}\sin x$

9. $y = c_1\cos 4x + c_2\sin 4x - (26/4913)e^x - (4/289)xe^x + (1/17)x^2e^x$

11. $y = c_1e^{-2x} + c_2e^{-x} + 3e^{-2x}\ln(1+e^x) + 3e^{-x}\ln(1+e^x)$

13. $y = c_1\cos x + c_2\sin x - 1 - \cos x + 2\mathrm{Arctanh}[\sin x/(1+\cos x)]\sin x$

15. $y = c_1x + c_2/x^3 - (4/7)\sqrt{x}$

17. $y = c_1(2x^2-1) + c_2x(x^2-1)^{1/2} + x/3$

19. $y = x^3 - 2x^2\ln x - x$

21. $y = 2\cos x - 2 + 4\mathrm{Arctanh}\left(\dfrac{\sin x}{1+\cos x}\right)\sin x$

23. $y_1(x) = x$, $y_2(x) = 1-x$, $W(t) = -1$,

$$G(x,t) = \begin{cases} t(x-1), & 0 \le t < x \\ x(t-x), & x < t \le 1 \end{cases}$$

25. $y_1(x) = \sin\lambda x$, $y_2(x) = \dfrac{\sin\lambda(1-x)}{\sin\lambda}$, $W(t) = -\lambda$,

$$G(x,t) = \begin{cases} \dfrac{\sin\lambda t\sin\lambda(x-1)}{\lambda\sin\lambda}, & 0 \le t < x \\ \dfrac{\sin\lambda x\sin\lambda(t-1)}{\lambda\sin\lambda}, & x < t \le 1 \end{cases}$$

27. $y_1(x) = x - 1/x$, $y_2(x) = x - 4/x$, $W(t) = 6/t$, $y(x) = \dfrac{e^{-x}}{x}(1+x) + \dfrac{1}{xe^2}(1-x^2) + \dfrac{2}{3xe}(x^2-4)$

29. $y_1(x) = 3x - x^3$, $y_2(x) = 4x - x^3$, $W(t) = 2t^3$, $y(x) = x(4 - x\ln x - x^2) - 2x\ln 2(3-x^2)$

Exercise Set 6.7

1. $y_2 = e^{-2x}$

3. $y_2 = e^{-x}\sin x$

5. $y_2 = x\ln|x|$

7. $y_2 = (1/x)\cos x$

9. $y_2 = \ln|x|$

Exercise Set 6.8

1. $u'' + \left(\dfrac{9}{x} - \dfrac{1}{4x^2}\right)u = 0$ **3.** $y = c_1e^x + c_2xe^{-x}$

5. $y = e^{2x}(c_1\cos x + c_2\sin x)$

7. $y = c_1(1/x)\sin x + c_2(1/x)\cos x$

Exercise Set 6.9

1. $x_1 = c_1e^{2t} - c_2e^t$, $x_2 = -3c_1e^{2t} + c_2e^t$

3. $x_1 = -6e^{2t} + 6e^{-t}$, $x_2 = 4e^{2t} - 3e^{-t}$

5. $x_1 = (5/3) - 4e^t + 9e^{2t} - (25/6)e^{3t} - (3/2)e^{-t}$, $x_2 = -(4/3) + 2e^t - 3e^{2t} + (25/12)e^{3t} + (1/4)e^{-t}$, $x_3 = -(1/2)e^{-t} + 2/3 - 2e^t + 6e^{2t} - (25/6)e^{3t}$

Exercise Set 6.10

1. $\Phi(t) = \begin{bmatrix} e^t\cos t & e^t\sin t \\ -e^t\sin t & e^t\cos t \end{bmatrix}$

3. $\Phi(t) = \begin{bmatrix} \cos t & \sin t \\ \frac{1}{2}(\cos t + \sin t) & \frac{1}{2}(\sin t - \cos t) \end{bmatrix}$

5. $\Phi(t) = \begin{bmatrix} e^{3t/2}\cos\frac{1}{2}t & e^{3t/2}\sin\frac{1}{2}t \\ e^{3t/2}\left(\sin\frac{1}{2}t - \cos\frac{1}{2}t\right) & -e^{3t/2}\left(\cos\frac{1}{2}t + \sin\frac{1}{2}t\right) \end{bmatrix}$

Exercise Set 6.11

1. $\Phi(t) = \begin{bmatrix} \sin t\sqrt{2} & \cos t\sqrt{2} \\ -\sqrt{2}\cos t\sqrt{2} & \sqrt{2}\sin t\sqrt{2} \end{bmatrix}$; $x_1(t) = C_1\sin t\sqrt{2} + C_2\cos t\sqrt{2}$, $x_2(t) = C_2\sqrt{2}\sin t\sqrt{2} - C_1\sqrt{2}\cos t\sqrt{2}$

3. $\Phi(t) = \begin{bmatrix} -2e^{-t}\sin 2t & e^{-t}(\cos 2t - 2\sin 2t) \\ e^{-t}(\sin 2t + \cos 2t) & e^{-t}\sin 2t \end{bmatrix}$; $x_1(t) = -(2C_1 + C_2)e^{-t}\sin 2t + C_2e^{-t}\cos 2t$, $x_2(t) = (C_1 + C_2)e^{-t}\sin 2t + C_1e^{-t}\cos 2t$

5. $\Phi(t) = \begin{bmatrix} 1 & \sin 2t & \cos 2t \\ 0 & \cos 2t & -\sin 2t \\ 0 & \sin 2t & \cos 2t \end{bmatrix}$; $x_1(t) = C_1 + C_2\sin 2t + C_3\cos 2t$, $x_2(t) = -C_3\sin 2t + C_2\cos 2t$, $x_3(t) = C_2\sin 2t + C_3\cos 2t$

7. $x_1(t) = \frac{95}{4} + \frac{11}{2}t - \frac{3}{2}C_1e^{2t} - 2C_2e^{-t}$, $x_2(t) = -\frac{27}{2} - 3t + C_1e^{2t} + C_2e^{-t}$

9. $x_1(t) = -(1/5)\cos t + (3/5)\sin t - (1/3)e^{3t} + C_1 + C_2e^{2t}$ $x_2(t) = (2/3)e^{3t} + (2/5)\sin t + (1/5)\cos t + C_1 - C_2e^{2t}$

11. $x_1(t) = (1/8)\cos t + (1/4)\sin t + C_1e^{t\sqrt{7}} + C_2e^{-t\sqrt{7}}$ $x_2(t) = (1/8)\sin t + (1/3)C_1(\sqrt{7}-2)e^{t\sqrt{7}} - (1/3)C_2(\sqrt{7}+2)e^{-t\sqrt{7}}$

13. $x_1(t) = -3 - (3/5)\cos t + (4/5)\sin t + C_2e^{2t} + 2(C_1 + C_3)e^{-t}$, $x_2(t) = 3 + C_1e^{-t}$ $x_3(t) = -6 - (1/5)\cos t + (3/5)\sin t + 2C_2e^{2t} + C_3e^{-t}$

15. $x_1(t) = -3/5 - t - 2C_1e^{-t} + (C_3 - C_2)e^{2t}\sin t + C_2e^{2t}\cos t$ $x_2(t) = -4/5 + C_1e^{-t} - C_2e^{2t}\sin t + (C_3 - C_2)e^{2t}\cos t$ $x_3(t) = 6/5 + 2C_1e^{-t} + C_3e^{2t}\sin t + (2C_2 - C_3)e^{2t}\cos t$

17. $x_1(t) = -4/3 - e^{-t} + 2C_1e^{3t} + (C_2 - C_3)\sin t$
$+ C_2 \cos t$
$x_2(t) = 1/3 + t - (1/2)e^{-t} + C_1e^{3t} - C_2 \sin t$
$+ (C_2 - C_3)\cos t$
$x_3(t) = 2/3 - 2t + e^{-t} + 2C_1e^{3t} + C_3 \sin t$
$+ (C_3 - 2C_2)\cos t$

19. $x_1(t) = C_1e^{2t} + C_2e^{t}, \quad x_2(t) = -3C_1e^{2t} - C_2e^{t}$

21. $x_1(t) = C_1 \sin t\sqrt{2} + C_2 \cos t\sqrt{2},$
$x_2(t) = C_2\sqrt{2}\sin t\sqrt{2} - C_1\sqrt{2}\cos t\sqrt{2}$

23. $x_1(t) = (4C_2 - 17C_1)e^{-t}\sin 2t + C_2e^{-t}\cos 2t$
$x_2(t) = (C_2 - 4C_1)e^{-t}\sin 2t + C_1e^{-t}\cos 2t$

25. $x_1(t) = -(2C_1 + C_2)e^{-t}\sin 2t + C_2e^{-t}\cos 2t,$
$x_2(t) = (C_1 + C_2)e^{-t}\sin 2t + C_1e^{-t}\cos 2t$

27. $x_1(t) = -(7/5)\cos t - (16/5)\sin t - 9t - 9/2$
$- 2C_1e^{t} - (3/2)C_2e^{-2t}$
$x_2(t) = (3/5)\cos t + (9/5)\sin t + 2 + 5t + C_1e^{t}$
$+ C_2e^{-2t}$

29. $x_1(t) = -(4/5)t^2 - (16/25)t + 8/125 + (2C_1 + C_2)$
$e^{t}\sin 2t + C_2e^{t}\cos 2t$
$x_2(t) = (2/25)t - 26/125 + (3/5)t^2 - (C_1 + C_2)$
$e^{t}\sin 2t + C_1e^{t}\cos 2t$

31. $x_1(t) = -(3/4) - (1/2)t + (5/3)e^{t} + (1/12)e^{-2t},$
$x_2(t) = -3/2 + (5/3)e^{t} - (1/6)e^{-2t}$

33. $x_1(t) = 3t - e^{t} + 1 - 2te^{t},$
$x_2(t) = -6t + 1 + 2te^{t}$

35. $x_1(t) = -5/2 + (1/10)\cos t + (3/10)\sin t + 2e^{2t}$
$- (61/10)e^{-2t} + (15/2)e^{-t}$
$x_2(t) = -15/2 + 6t - (1/5)\sin t + (3/5)\cos t$
$- 2e^{2t} - (61/10)e^{-2t} + 15e^{-t}$
$x_3(t) = 15/4 - (5/2)t + (1/10)\sin t - (3/10)$
$\cos t + (61/20)e^{-2t} - (15/2)e^{-t}$

Exercise Set 6.12

1. Saddle point at (0, 0)

3. Stable focus at (0, 0)

5. Stable focus at $(\frac{46}{13}, \frac{2}{13})$

7. Saddle point at $(-2, 0)$ and an unstable node at $(2, 0)$

9. Saddle point at (0, 0) and linear theory predicts a center at $(\frac{1}{4}, -\frac{1}{2})$. An examination of the phase portrait shows that the point $(\frac{1}{4}, -\frac{1}{2})$ is also a center of the nonlinear system.

11. For $\varepsilon \le -2$, the point (0, 0) is a stable node.
For $-2 < \varepsilon < 0$, the point (0, 0) is a stable focus.
For $0 < \varepsilon < 2$, the point (0, 0) is an unstable focus.
For $\varepsilon \ge 2$, the point (0, 0) is an unstable node.

Exercise Set 7.1

5. $1/(s-2)^2$ **9.** $s/(s^2 + 4s + 8)$

7. $1/s^2 - 2/s^3 + 6/s^4$ **11.** $1/(5+3)^2$

13. $(s^3 - 2s - 5)/[s^2(s^2 + 2s + 5)]$

15. $e^{\pi/2}e^{-\pi s/2}(s-1)/(s^2 - 2s + 2)$

17. $\pi e^{-\pi s/2}/(2s) + e^{-\pi s/2}/s^2 - \pi e^{-\pi s}/s - e^{-\pi s}/s^2$

19. $-e^{-\pi/2}e^{-\pi s/2}/(s^2 + 2s + 2)$

21. $-1/4 + (5/4)\cos 2t$

23. $5/9 + \sin 3t - (5/9)\cos 3t$

25. $(9/5)te^{-2t} - (96/25)e^{-2t} + (13/75)e^{3t} + (14/3)e^{-3t}$

27. $(1/4)e^{-t} + (1/2)te^{t} + (3/4)e^{t}$

29. $-(5/8)e^{t} + (13/12)e^{3t} + (13/24)e^{-3t}$

31. $F(s) = 1/s + (e^{-2as} - 2e^{-as})/s$

33. $F(s) = k/s^2 - ke^{-s}/s^2$

35. $F(s) = k(1 + e^{-2as} - 2e^{-as})/as^2$

Exercise Set 7.2

3. $s^3F(s) - s^2 - 1$ **5.** $(1 - se^{-\pi s/2})/(s^2 + 1)$

7. $(1/10)\cos t - (3/10)\sin t + (5/2)e^{t} - (8/5)e^{2t}$

9. $-(8/81) - (1/9)t + 2e^{t} + (8/81)e^{-9t}$

11. $2/(s+2) + 6/(s+2)^4$

13. $(4s+4)/(s^2 + 2s + 5)^2$

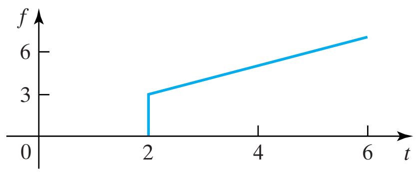
15. $3/[s^2 - 4s + 13]$ **17.** $(1/3)e^{2t}\sin 3t$

19. $e^{-t}(2\sin 2t - 3\cos 2t)$

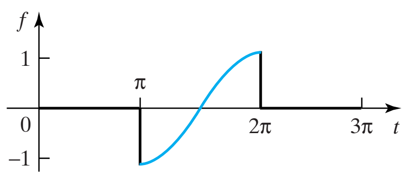
21. $-(1/18)e^{-t} + e^{2t}[(1/18)\cos 3t + (5/18)\sin 3t]$

23. $3/2 + e^{-2t}[(3/2)\cos 2t - (9/2)\sin 2t]$

25.



27.



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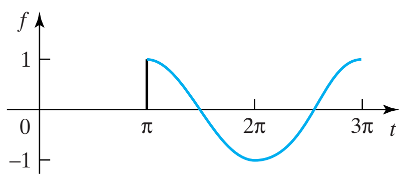

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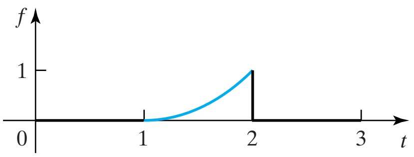


33. $6e^{-3s}/s^4$ **37.** $3e^{-4s}/(s^2-9)$

35. $2e^{-3\pi s/2}/(4+s^2)$ **39.** $H(t-2)\cos(2t-4)$

41. $H(t-\pi/2)e^{\pi-2t}(\cos t+\sin t)$

43. $H(t-4)e^{8-2t}\{(1/3)\sin(3t-12)+\cos(3t-12)\}$

45. $y(t)=3e^{-2t}-2e^{-3t}+(1/10)(3e^{3(\pi-t)}-4e^{2(\pi-1)}-\cos t-\sin t)H(t-\pi)$

47. $y(t)=-(3/2)e^{2t}+(4/3)e^{3t}+1/6+(1/36)(5+6t-27e^{2t-4}+28e^{3t-6})H(t-2)$

49. $y(t)=-e^{-t}\cos 3t-(1/3)e^{-t}\sin 3t+(1/9)e^{-t}(1-\cos(3t-3))H(t-1)$

51. $2(3s^2-18s+26)/(s^2-6s+10)^3$

53. $48s(s-2)(s-4)/(s^2-4s+8)^4$

55. $2e^{-3s/2}(s^2-4)/(s^4-16a^4)$

57. $1/(27s^4+12s^2)$ **59.** $1/(s+se^{-ks})$

61. $(1/s^2)\tanh ks$ **63.** $k/(as^2)-ke^{-as}/(s-se^{-as})$

65. $k(1-2ase^{-as}-e^{-2as})/[as^2(1-e^{-2as})]$

67. $e^{-t}-e^{-2t}$

69. $t^2+2\cos t-2$

71. $(1/2)(\sin t+t\cos t)$

73. $1/[s^2(s+2)]$

75. $1/[s^2(s^2+2s+2)]$

77. $(1/4)t-(1/8)\sin 2t$

79. $(1/2)t\cosh t+(1/2)\sinh t$

81. $y(t)=t$

83. $y(t)=t^2+2t+2-e^{t/2}\{(2/\sqrt{3})\sin(t\sqrt{3}/2)+2\cos(t\sqrt{3}/2)\}$

85. $y(t)=1-(4/\sqrt{3})e^{-2t}\sinh t\sqrt{3}$

87. $y(t)=(1/2)(1+\cosh t\sqrt{2})$

89. $(12s^2-16)/[s(s^2+4)^3]$

91. $(\sin at-at\cos at)/(2a^3)$

93. $(1/2)\ln\{(s+2)/(s-2)\}$

95. $(2/t)(1-\cosh at)$

97. $f(t)=3/2+(1/2)\cos 3t$; $f(0)=1,\ f'(0)=0,\ f''(0)=-3$

99. $f(t)=e^{2t}(1+t)$; $f(0)=1,\ f'(0)=3,\ f''(0)=8$

101. $-4/\pi$

103. $-8/(21\pi)$

105. $y(t)=(2/9)\sin^2(3t/2)+(1/3)\sin(3t-3)H(t-1)$

107. $y(t)=(1/2)e^{-t}(1+t)-(1/2)\cos t+(t-\pi)e^{\pi-t}H(t-\pi)$

109. $y(t)=1/2+\cos 2t-(1/2)\cos^2 t-(1/2)(1-\cos^2(t-1))H(t-1)+(1/2)\sin(2t-4)H(t-2)$

Exercise Set 7.3

1. $x(t)=27/49+(8/7)t-(27/49)e^{-7t}$,
$y(t)=71/49+(20/7)t+(27/49)e^{-7t}$

3. $x(t)=3/2+\sqrt{2}\sinh t\sqrt{2}-(5/2)\cosh t\sqrt{2}$,
$y(t)=1/2+(3/2)\sinh t\sqrt{2}+(1/2)\cosh t\sqrt{2}$

5. $x(t)=5/2+(1/2)t+e^t\{3^{3/2}\sinh t\sqrt{3}-(1/2)\cosh t\sqrt{3}\}$
$y(t)=1+(1/2)t+e^t\{(1/6)\sqrt{3}\sinh t\sqrt{3}-3\cosh t\sqrt{3}\}$

7. $x(t)=7/8+(5/4)t-(1/4)t^2+(1/8)e^{-2t}$,
$y(t)=1/8+(7/4)t-(1/4)t^2-(1/8)e^{-2t}$
$z(t)=9/8+(3/4)t-(1/4)t^2-(1/8)e^{-2t}$

9. $x(t)=-1+(1/4)e^{-t}+(1/4)e^t(3+2t)$,
$y(t)=1+2t+(1/4)e^{-t}+(1/4)e^t(2t-1)$,
$z(t)=-(1/4)e^{-t}+(1/4)e^t(1+2t)$

11. $\begin{bmatrix}\frac{1}{4}e^{-2t}+\frac{3}{4}e^{2t} & \frac{3}{4}e^{2t}-\frac{3}{4}e^{-2t}\\ \frac{1}{4}e^{2t}-\frac{1}{4}e^{-2t} & \frac{3}{4}e^{-2t}+\frac{1}{4}e^{2t}\end{bmatrix}$

13. $\begin{bmatrix}\frac{1}{4}e^{-3t}+\frac{3}{4}e^{5t} & \frac{3}{4}e^{5t}-\frac{3}{4}e^{-3t}\\ \frac{1}{4}e^{5t}-\frac{1}{4}e^{-3t} & \frac{3}{4}e^{-3t}+\frac{1}{4}e^{5t}\end{bmatrix}$

15. $\begin{bmatrix}e^{2t}\cos 4t+\frac{1}{2}e^{2t}\sin 4t & -\frac{5}{4}e^{2t}\sin 4t\\ e^{2t}\sin 4t & e^{2t}\cos 4t-\frac{1}{2}e^{2t}\sin 4t\end{bmatrix}$

17. $\begin{bmatrix}e^{2t}\cos 2t & -2e^{2t}\sin 2t\\ \frac{1}{2}e^{2t}\sin 2t & e^{2t}\cos 2t\end{bmatrix}$ **19.** $\begin{bmatrix}e^{6t} & -te^{6t}\\ 0 & e^{6t}\end{bmatrix}$

21. $\begin{bmatrix}e^{-2t} & 4te^{-2t}\\ 0 & e^{-2t}\end{bmatrix}$

23. $\begin{bmatrix}e^{5t} & \frac{11}{5}e^{5t}-e^t-\frac{6}{5} & \frac{8}{5}e^{5t}-e^t-\frac{3}{5}\\ 0 & 2-e^t & 1-e^t\\ 0 & 2e^t-2 & 2e^t-1\end{bmatrix}$

27. $y(t)=(1/10)e^{2t}-(3/10)e^{-t}\sin t-(1/10)e^{-t}\cos t$,
$W(t)=e^{-t}\sin t$

29. $y(t)=(1/16)e^{-5t}+(1/4)te^{-t}-(1/16)e^{-t}$,
$W(t)=(1/4)(e^{-t}-e^{-5t})$

31. $x(t)=-1-(5/14)e^{-t}-(1/7)e^{6t}+(3/2)e^t$
$y(t)=-3/2-(3/14)e^{-t}+(3/14)e^{6t}+(3/2)e^t$

39. $x(t)=\sin t-(1/3)\sin 2t$

41. $y(x) = \dfrac{M}{24aEI}x^4 + \dfrac{Q}{6EI}(x-3a/4)^3 H(x-3a/4) + \dfrac{a}{384EI}(16M+9Q)x^2 - \dfrac{Q}{192EI}(16M+5Q)x^3$; $w(x) = M/a + Q\delta(x-3a/4)$

43. $$i(t) = \frac{E_0 C}{\sqrt{R^2C^2-4LC}}\exp\left(-\frac{Rt}{2L}\right) \times \left\{\exp\left(\frac{t}{2}\frac{\sqrt{R^2C^2-4LC}}{LC}\right) - \exp\left(-\frac{t}{2}\frac{\sqrt{R^2C^2-4LC}}{LC}\right)\right\}$$

The solution is oscillatory if $4L > R^2C$; otherwise it behaves exponentially.

45. $x(t) = Qe^{-2t}$, $y(t) = 2Q(e^{-2t} - e^{-3t})$, $z(t) = 6Q(e^{-2t} - e^{-3t} - te^{-3t})$ so $w(t) = Q(1 - 9e^{-2t} + 8e^{-3t} + 6te^{-3t})$. After 1, 2, and 3 time units $w(t)/Q = 48\%, 88\%$, and 98%, respectively.

Exercise Set 7.4

1. **(a)** Order 3, roots $s = 1, s = -2 \pm 4i$, unstable
(b) Order 3, roots $s = -2, s = -1 \pm 3i$, stable
(c) Order 2, roots $s = -\frac{1}{3} \pm i$, stable

Exercise Set 8.1

1. $y(x) = 1 - x + (1/2)x^2 - (1/6)x^3 + (7/24)x^4 - (19/120)x^5 + \cdots$

3. $y(x) = -1 + x - x^2 + x^3 - (3/4)x^4 + (11/20)x^5 + \cdots$

5. $y(x) = 1 + x - (1/2)x^2 + (1/3)x^3 + (5/8)x^4 - (4/15)x^5 + \cdots$

7. $y(x) = 2 - (1/3)x + (1/18)x^2 + (35/162)x^3 - (89/1944)x^4 + (197/29160)x^5 + \cdots$

9. $y(x) = a + bx + (1/3)bx^3 - (1/12)ax^4 + (1/20)bx^5 - (1/45)ax^6 + (1/252)bx^7 + \cdots$

11. $y(x) = a + bx + \{-(1/2)a + 1/2\}x^2 + \{-(2/3)b + 1/6\}x^3 + \{(11/24)a - 3/8\}x^4 + \cdots$

13. $y(x) = a + bx - (1/6)ax^3 - (1/12)bx^4 + (1/180)ax^6 + (1/504)bx^7 + \ldots$

15. $y(x) = a + bx - (1/4)ax^2 + (1/12)(2-b)x^3 + (1/96)(5a - 12b)x^4 + \ldots$

Exercise Set 8.2

1. $y(x) = 2 - 3x - x^2 + x^3 - (3/10)x^5 + (1/10)x^6 + \cdots$

3. $y(x) = 1 - 3x + (3/2)x^2 - (2/3)x^3 + (2/3)x^4 - (43/120)x^5 + \cdots$

5. $y(x) = 2 - x + x^2 + (1/12)x^4 + (1/40)x^6 + \cdots$

7. $y(x) = 1 - x + x^2 - (1/2)x^3 + (1/3)x^4 - (2/15)x^5 + \cdots$

9. $y(x) = 1 - x - (1/2)x^2 + (5/6)x^3 - (11/24)x^4 + (67/120)x^5 + \cdots$

11. $y(x) = 1 + 4x + 3x^2 + 3x^3 + (11/4)x^4 + (31/10)x^5 + \cdots$

13. $y(x) = 2 - 3(x-1) + (7/3)(x-1)^2 - (53/54)(x-1)^3 + (11/81)(x-1)^4 + (319/3240)(x-1)^5 + \cdots$

15. $y(x) = 1 + 5(x-2) + 8(x-2)^2 + 6(x-2)^3 + (13/6)(x-2)^4 + (7/30)(x-2)^5 + \cdots$

17. Proceed as outlined in the exercise

19. Proceed as outlined in the exercise

Exercise Set 8.3

1. Regular singular point at $x = 1$

3. Irregular singular point at $x = -1$

5. Irregular singular point at $x = -4$

7. Irregular singular point at $x = 3$

Exercise Set 8.4

1. **(a)** $a_0x^{c-2} + (a_0 + a_1)x^{c-1} + \sum_{n=0}^{\infty}(2a_n + a_{n+1} + a_{n+2})x^{n+c}$
(b) $3a_0x^c + \sum_{n=0}^{\infty}(2a_n + 3a_{n+1})x^{n+c+1}$

3. **(a)** $1 + (1/2)x - (1/12)x^2 + (1/24)x^3 - (9/720)x^4 + \cdots$
(b) $1 - (1/4)x^2 - (5/24)x^3 - (1/16)x^4 - (11/480)x^5 - \cdots$
(c) $1 - (3/2)x + (4/3)x^2 - (7/6)x^3 + (31/30)x^4 + \cdots$

5. **(a)** $\ln x - 2x - (1/4)x^2 - (4/9)x^3 - (15/32)x^4 + \cdots +$ constant
(b) $\ln x - (1/4)x^2 + (2/9)x^3 - (1/32)x^4 - (8/75)x^5 + \cdots +$ constant
(Hint: write the integrand as $\frac{1}{x}\frac{e^x}{(1+x+x^2)}$)

7. $c = 1$, $y_1(x) = x\{1 - (1/10)x + (1/280)x^2 - (1/15120)x^3 + \cdots\}$;
$c = -1/2$, $y_2(x) = x^{-1/2}\{1 + (1/2)x - (1/8)x^2 + (1/144)x^3 + \cdots\}$

9. $c=1$, $y_1(x)=x\{1+(2/5)x+(2/35)x^2+(4/945)x^3+\cdots\}$;
$c=-1/2$, $y_2(x)=x^{-1/2}\{1-2x-2x^2-(4/9)x^3-(2/45)x^4+\cdots\}$

11. $y_1(x)=1+\frac{1}{2!}x^2+\frac{7}{4!}x^4+\frac{49}{240}x^6+\cdots$, $y_2(x)=x+\frac{1}{2}x^2+\frac{13}{40}x^5+\frac{403}{1680}x^7+\ldots$

13. $y_1(x)=1+x+\frac{2}{4}x^2+\frac{2\cdot 5}{4\cdot 9}x^3+\frac{2\cdot 5\cdot 10}{4\cdot 9\cdot 16}x^4+\cdots$
$y_2(x)=y_1(x)\ln x-2x-x^2-(14/27)x^3-\cdots$

15. $c=1$ (twice), $y_1(x)=xe^{-2x}$; $c=1$, $y_2(x)=y_1(x)\{\ln x+2x+x^2+(4/9)x^3+\cdots\}$

17. $c=2$, $y_1(x)=x^2e^{-x}$; $c=1$, $y_2(x)=y_1(x)\{\ln x-1/x+(1/2)x+(1/12)x^2+\cdots\}$

19. $c=1/4$ (twice), $y_1(x)$
$$=x^{1/4}\left\{1-x+\frac{1}{2^2}x^2-\frac{1}{2^23^2}x^3+\frac{1}{2^23^24^2}x^4+\cdots\right\}$$
$c=1/4$, $y_2(x)=y_1(x)\{\ln x+2x+(5/4)x^2+(23/27)x^3+\cdots\}$

21. $c=3$, $y_1(x)=x^3\{1-(3/5)x+(1/5)x^2-(1/21)x^3+(1/112)x^4+\cdots\}$;
$c=-1$, $y_2(x)=x^{-1}\{1-(1/3)x\}$

23. $c=2$, $y_1(x)=x^2(1-(2/5)x+(1/10)x^2-(2/105)x^3+\cdots)$; $c=-2$, $y_2(x)=y_1(x)$
$$\left[\frac{1}{168}\ln x-\frac{1}{4x^4}+\frac{1}{15x^3}+\frac{1}{100x^2}-\frac{13}{1750x}+\cdots\right]$$

25. $c=2\pm 4i$, $y_1(x)=x^2\cos(4\ln|x|)$; $y_2(x)=x^2\sin(4\ln|x|)$

27. Shift the critical point at $x=-1$ to the origin by setting $X=x+1$ and solve the resulting equation to get
$$c=1,\quad y_1(X)=1+\frac{1}{2\cdot 3}X^2+\frac{1}{(2\cdot 4)(3\cdot 7)}X^4+\frac{1}{(2\cdot 4\cdot 6)(3\cdot 7\cdot 9)}X^6+\cdots$$
and
$c=1/2$, $y_2(X)$
$$=X^{1/2}\left(1+\frac{1}{2\cdot 5}X^2+\frac{1}{(2\cdot 4)(5\cdot 9)}X^4+\cdots\right)$$
The required results follows by substituting $X=x+1$. The results converge in an interval of the form $0<x+1<d$ for some suitable d.

Exercise Set 8.5

1. $\Gamma(5/2)=(3/4)\sqrt{\pi}$, $\Gamma(-5/2)=-(8/15)\sqrt{\pi}$, $\Gamma(9/2)=(105/16)\sqrt{\pi}$

3. $\Gamma(5/4)=(1/4)\Gamma(1/4)$, $\Gamma(-5/4)=-(4/5)\Gamma(-1/4)$, $\Gamma(7/4)=-(3/16)\Gamma(-1/4)$

5. $5^{n+1}\Gamma(6/5+n+1)/\Gamma(6/5)$

7. $3^{n+1}\Gamma(8/3+n)/\Gamma(5/3)$

9. $(\frac{1}{2}-n)\Gamma(\frac{1}{2}-n)=\Gamma(\frac{3}{2}-n)$, so $\Gamma(\frac{1}{2}-n)=-\Gamma(\frac{3}{2}-n)/(n-\frac{1}{2})$, similarly, $(\frac{3}{2}-n)\Gamma(\frac{3}{2}-n)=\Gamma(\frac{5}{2}-n)$, so $\Gamma(\frac{3}{2}-n)=-\Gamma(\frac{5}{2}-n)/(n-\frac{3}{2})$ giving $\Gamma(\frac{1}{2}-n)=(-1)^2\Gamma(\frac{5}{2}-n)/(n-\frac{1}{2})\times(n-\frac{3}{2})$. Continuing this process leads to $\Gamma(\frac{1}{2}-n)=(-1)^n\Gamma(\frac{1}{2})/(n-\frac{1}{2})(n-\frac{3}{2})\ldots(\frac{1}{2})=(-1)^n\sqrt{\pi}/(n-\frac{1}{2})(n-\frac{3}{2})\ldots(\frac{1}{2})$

11. $\Gamma(2n)$
$$\begin{aligned}&=(2n-1)!=(2n-1)(2n-2)\ldots 3\cdot 2\cdot 1\\&=2^{2n-1}(n-\tfrac{1}{2})(n-1)(n-\tfrac{3}{2})\ldots(\tfrac{3}{2})\cdot 1\\&=2^{2n-1}\{(n-\tfrac{1}{2})(n-\tfrac{3}{2})\ldots(\tfrac{1}{2})\}\\&\quad\times\{(n-1)(n-2)\ldots 2\cdot 1\}\\&=2^{n-1}\{(n-\tfrac{1}{2})(n-\tfrac{3}{2})\ldots(\tfrac{1}{2})\}\Gamma(n)\\&=2^{n-1}\{(n-\tfrac{1}{2})(n-\tfrac{3}{2})\ldots(\tfrac{1}{2})\Gamma(\tfrac{1}{2})\}\Gamma(n)/\Gamma(\tfrac{1}{2})\\&=2^{n-1}\Gamma(n+\tfrac{1}{2})\Gamma(n)/\sqrt{\pi}\end{aligned}$$

13. Make the substitution $t=u^2$ in the definition of $\Gamma(x)$ in (32).

15. $\psi(x+n)=d/dx\{\ln\Gamma(x+n)\}=d/dx\{\ln[(x+n-1)\Gamma(x+n-1)]\}=1/(x+n-1)+d/dx\{\ln\Gamma(x+n-1)\}$ a repetition of this argument leads to $\psi(x+n)=1/(x+n-1)+1/(x+n-2)+\cdots 1/x+\psi(x)=\Sigma_{k=0}^{n-1}1/(x+k)+\psi(x)$

17. The result follows directly after integrating by parts.

Exercise Set 8.6

1. $J_2(x)=(1/8)x^2-(1/96)x^4+(1/3072)x^6-(1/184320)x^8+(1/17694720)x^{10}-(1/2477260800)x^{12}+\cdots$

5. 6 terms **7.** 6 terms **9.** 6 terms

11. $(1/4)x^2-(1/64)x^4+(1/2304)x^6-(1/147456)x^8$; max magnitude of error is $a^{10}/14745600$

12 to **17.** If $x=\lambda X$, then $d/dx=(dX/dx)d/dX=(1/X)d/dx$. Substitute $x=\lambda X$ and use results (64)–(67).

19. The first two limits follow from the series for $J_\nu(x)$ in (54). The third follows by taking the limit as $x\to\infty$ in result (70):
$$\int_0^\infty J_1(x)dx=-\int_0^\infty J_0'(x)dx=[-J_0(x)]_0^\infty=1.$$

21. $\mathcal{L}\{J_0(x)\}=\int_0^\infty e^{-xs}J_0(x)dx=1/(s^2+1)^{1/2}$. Setting $s=0$ gives $\int_0^\infty J_0(x)dx=1$. From (67)

with $\nu = 2n+1$ we have $\int_0^\infty J_{2n}(x)dx - \int_0^\infty J_{2n+2}(x)dx = 2[J_{2n+1}(x)]_0^\infty = 0$.
As $\int_0^\infty J_0(x)dx = 1$ we have $1 = \int_0^\infty J_0(x)dx = \int_0^\infty J_1(x)dx = \int_0^\infty J_2(x)dx = \cdots$

23. $\int J_4(x)dx = -2J_1(x) - 2J_3(x) + \int J_0(x)dx$

25. $\int xJ_1(x)dx = -xJ_0(x) + \int J_0(x)dx$

Exercise Set 8.7

1. $y(x) = C_1 J_2(x) + C_2 Y_2(x)$

3. $y(x) = C_1 J_0(x) + C_2 Y_0(x)$

5. $y(x) = C_1 J_0(x^2) + C_2 Y_0(x^2)$

7. $y(x) = C_1 J_2(2x) + C_2 Y_2(2x)$

9. $y(x) = x^{1/2}\{C_1 J_0(2x) + C_2 Y_0(2x)\}$

11. $a = 1, b = 1, c = 2, n = 1; y(x) = xZ_1(x^2)$

13. $a = 1, b = 3, c = 1, n = 0; y(x) = xZ_0(3x)$

15. $a = 2, b = 2, c = 4, n = 1; y(x) = x^3 Z_1(2x^4)$

17. For u to depend on J_0 and Y_0, we must set $a = 3$ and $\nu = 1$. Thus the general solution for u is $u(x) = AJ_0(kx) + BY_0(kx)$, so the general solution for y is $y(x) = (1/x)(AJ_0(kx) + BY_0(kx))$.

Exercise Set 8.8

5. Replacing $\sinh x$ and $\cosh x$ by their definitions in terms of exponentials and comparing with (106) shows that $C_1 = C_2 = \sqrt{(2/\pi)}$, so

$$I_{1/2}(x) = \sqrt{2/\pi x}\sinh x \quad \text{and}$$
$$I_{-1/2}(x) = \sqrt{2/\pi x}\cosh x.$$

Using this with the result of Exercise 2 gives

$$I_{3/2}(x) = -\sqrt{\frac{2}{\pi x}}\left(\frac{\sinh x}{x} - \cosh x\right) \text{ and}$$
$$I_{-3/2}(x) = -\sqrt{\frac{2}{\pi x}}\left(\frac{\cosh x}{x} - \sinh x\right).$$

7. Replace x by ix in $J_{\pm 1/2}(x)$ and $J_{\pm 3/2}(x)$ and remove any multiplicative factors i to obtain the results of Exercise 5.

9. Substituting the series for $I_\nu(x)$ and $I_{-\nu}(x)$ into the expression on the left of Exercise 8 shows that C, the coefficient of the term in $(1/x)$, is given by $C = -2\nu/\{\Gamma(1+\nu)\Gamma(1-\nu)\}$. Using $\Gamma(1+\nu) = \nu\Gamma(\nu)$ and the result $\Gamma(\nu)\Gamma(1-\nu) = \pi/\sin\pi\nu$ then gives $C = -(2/\pi)\sin\pi\nu$.

11. The expression $(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + 1)(\frac{d^2}{dr^2} + \frac{1}{r} - 1)R$ is equal to the left-hand side of the governing equation, so $\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + R = 0$ and $\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} - R = 0$ are both special solutions of the original fourth order equation. They have the respective solutions $R_1(r) = AJ_0(r) + BY_0(r)$ and $R_2(r) = CI_0(r) + DK_0(r)$, so the general solution of the original equation is $R(r) = R_1(r) + R_2(r)$. In a particular problem the initial conditions will determine the arbitrary constants A, B, C, and D.

Exercise Set 8.10

1. $\dfrac{d}{dx}[xe^{-x}y'] + \lambda e^{-x}y = 0$ (Laguerre's equation)

3. $\dfrac{d}{dx}[(1-x^2)^{1/2}y'] + \lambda(1-x^2)^{-1/2}y = 0$ (Chebyshev's equation)

5. $\lambda_n = n^2\pi^2/L^2, n = 1, 2, \ldots, \varphi_n = \sin\dfrac{n\pi x}{L}$

7. $\lambda_n = (2n-1)^2\pi^2/4, n = 1, 2, \ldots,$
$\varphi_n = \cos\dfrac{(2n-1)\pi x}{2}$

9. $\lambda_n = k_n^2$ where k_n are the roots of $\tan x = 2x$, $\varphi_n = \sin k_n x, \lambda_1 = k_1^2 \approx (1.166)^2 = 1.340, \lambda_2 = k_2^2 \approx (4.604)^2 = 21.197$

11. $\lambda_n = n^2\pi^2, n = 0, 1, \ldots,$
$\varphi_n = \{1, \cos n\pi x, \sin n\pi x\}$

13. General solution $y = C_1\cos(k\ln x) + C_2\sin(k\ln x)$, Eigenvalues $\lambda_n = k_n^2 = \left(\dfrac{n\pi}{2\ln 2}\right)^2$,
$\varphi_n = \sin\left(\dfrac{n\pi\ln x}{2\ln 2}\right)$

15. $\|\varphi_n\| = \sqrt{L/2}$ **16.** $\|\varphi_n\| = 1/\sqrt{2}$

17. $\|\varphi_0\| = \sqrt{L}, \|\varphi_n\| = \sqrt{L/2}, n = 1, 2, \ldots.$

19. An upper bound to λ_1 is

$$\int_0^\pi 4(\pi - x)^2dx \Big/ \int_0^\pi x^2(2\pi - x)^2dx = 5/2\pi^2 = 0.2533.$$

When Φ is substituted into the Rayleigh quotient, the constant C cancels.

21. An upper bound to λ_1 is

$$\left\{\left(\int_0^1 x(1-2x)^2dx + \int_0^1 x(1-x)^2dx\right)\Big/ \int_0^1 x^3(1-x)^2dx\right\} = 15,$$

so $j_{1,1} \approx \sqrt{15} = 3.87$.

Exercise Set 8.11

1. $(1/3)P_0(x) + (12/5)P_1(x) - (4/3)P_2(x) + (8/5)P_3(x)$

3. $(42/35)P_0(x) + 2P_1(x) + (18/7)P_2(x) + (8/35)P_4(x)$

5. $f(x) = (3/4)P_0(x) - (1/4)P_1(x) + (5/16)P_2(x) + (7/16)P_3(x) + \cdots$

7. $f(x) = (5/8)P_0(x) + (9/32)P_1(x) - (45/64)P_2(x) - (133/512)P_3(x) + \cdots$

9. $f(x) = (1/2)(e - 1/e)P_0(x) + (3/e)P_1(x) + (5/2)(e - 7/e)P_2(x) - (1/2)(35e - 259/e)P_3(x) + \cdots$

11. $-(7/8)T_0(x) - T_1(x) - (1/2)T_2(x) + (3/8)T_3(x)$

13. $(15/4)T_0(x) + (1/4)T_1(x) + T_2(x) - (1/4)T_3(x) + (1/4)T_4(x)$

15. $f(x) = (1/2\pi)(5\pi - 2)T_0(x) + (1/2\pi)(\pi + 4)T_1(x) - (2/3\pi)T_2(x) - (2/3\pi)T_3(x) + \cdots$

Exercise Set 9.1

1. 2π **3.** π **5.** 12π

7. $f(x)$ is not periodic **9.** $f(x)$ is not periodic

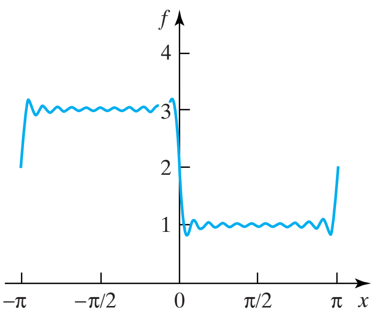
11. (a) $(1/2)\sin 2x$ (b) $\cos 2x$ (c) $(1/2)\sin 2x + (1/2)\sin 4x$

17. If $f(-x) = f(x)$ and $g(-x) = g(x)$ then $f(-x) + g(-x) = f(x) + g(x)$, so the sum is an even function. If $f(-x) = -f(x)$ and $g(-x) = -g(x)$, then $f(-x) + g(-x) = -f(x) - g(x)$, so the sum is an odd function.

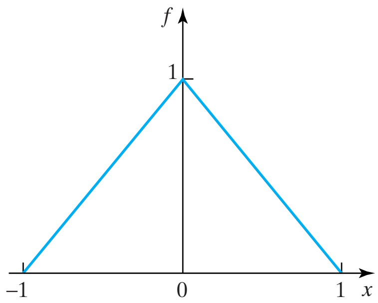
19. (a) $2L^2/\pi$ (b) $-L^2/\pi$ (c) $2L^2/3\pi$

23. $f(x) = \dfrac{a+b}{2} - \dfrac{2(a-b)}{\pi}\displaystyle\sum_{n=0}^{\infty}\dfrac{\sin(2n+1)x}{2n+1}$.

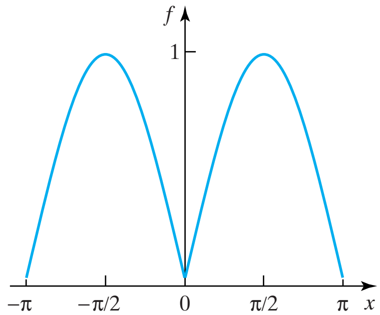
Graph for $a = 1, b = 3$.



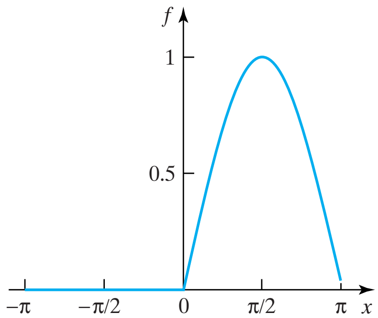
25. $f(x) = \dfrac{1}{2} + \dfrac{4}{\pi^2}\displaystyle\sum_{n=1}^{\infty}\dfrac{\cos(2n-1)\pi x}{(2n-1)^2}$



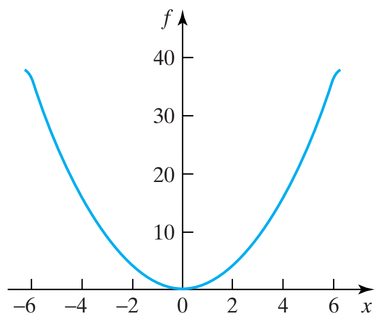
27. $f(x) = \dfrac{2}{\pi} - \dfrac{4}{\pi}\displaystyle\sum_{n=1}^{\infty}\dfrac{\cos 2nx}{4n^2-1}$



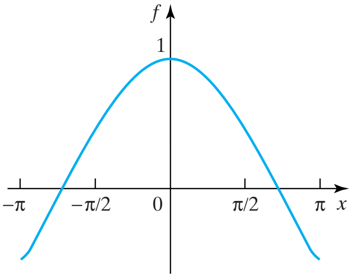
29. $f(x) = \dfrac{1}{\pi} + \dfrac{\sin x}{2} - \dfrac{2}{\pi}\displaystyle\sum_{n=1}^{\infty}\dfrac{\cos 2nx}{(4n^2-1)}$



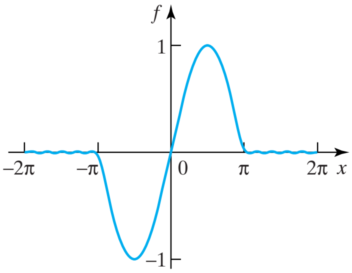
31. $f(x) = \dfrac{4\pi^2}{3} + 16\displaystyle\sum_{n=1}^{\infty}\dfrac{(-1)^n\cos\frac{1}{2}nx}{n^2}$



33. $f(x) = \dfrac{2\sin a\pi}{\pi}\left\{\dfrac{1}{2a} + \sum_{n=1}^{\infty}\dfrac{(-1)^n a\cos nx}{a^2-n^2}\right\}$.
Graph for $a = 0.7$, $n = 10$.



35. $f(x) = \dfrac{4}{3\pi}\sin\dfrac{1}{2}x + \dfrac{1}{2}\sin x$
$+\dfrac{4}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\dfrac{\sin\frac{1}{2}(2n+1)x}{(2n+1)^2-4}$



Exercise Set 9.2

1. $\dfrac{\pi^2}{8} = \sum_{n=1}^{\infty}\dfrac{1}{(2n-1)^2}$ **3.** $\dfrac{\pi^4}{90} = \sum_{n=1}^{\infty}\dfrac{1}{n^4}$

5. Proceed as in the derivation of the Parseval relation (27), but starting from the Fourier series representation of $f(x)$ on $-L \le x \le L$.

7. Set $x = 0$ with $f(0) = 0$ to get $\frac{\pi^2}{12} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}$

9. $f(x) = \dfrac{1}{2} - \dfrac{2}{\pi}\sum_{n=1}^{\infty}\dfrac{(-1)^n}{n}\sin\dfrac{1}{2}n\pi x$
$-\dfrac{4}{\pi^2}\sum_{n=1}^{\infty}\dfrac{\cos\frac{1}{2}(2n-1)\pi x}{(2n-1)^2}$.
Set $x = 0$ with $f(0) = 0$ to get $\frac{\pi^2}{8} = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$, or set $x = 2$ with $f(2) = 1$ for the same result.

11. The Fourier series for $f(x) = \pi^2 - x^2$ is $f(x) = \frac{2\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos nx}{n^2}$. As $f(-\pi) = f(\pi)$, Theorem 9.3 can be used to find the Fourier series for $f'(x)$ by differentiating term by term to obtain

$$x = 2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nx}{n}.$$

Theorem 9.2 can also be applied to obtain

$$x(\pi^2 - x^2) = 12\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nx}{n^3}.$$

13. Transform the result to

$$S_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(u)\left[\frac{1}{2} + \sum_{r=1}^{\infty}\cos[r(x-u)]\right]du.$$

Now set $t = x - u$ to obtain

$$S_n(x) = \frac{1}{\pi}\int_{x-\pi}^{x+\pi} f(x-t)\frac{\sin\left[\left(n+\frac{1}{2}\right)t\right]}{2\sin\frac{1}{2}t}dt.$$

17. $S_n(x) = \frac{1}{\pi}\int_0^{\pi}[f(x-t) + f(x+t)]D_n(t)dt$.
When n is large $D_n(t)$ can be replaced by $\Delta(t)$ to give

$$S_n(x) \approx \frac{(2n+1)}{4\pi}\int_0^{2\pi/(2n+1)} \times [f(x-t) + f(x+t)]dt,$$

and for large n the interval of integration is very small so the integrand is almost constant over the interval of integration, as a result of which integral can be replaced by

$$S_n(x) \approx \frac{(2n+1)}{4\pi}[f(x-t) + f(x+t)] \times \int_0^{2\pi/(2n+1)} dt = \frac{1}{2}[f(x-t) + f(x+t)],$$

and in the limit as $n \to \infty$ this becomes an equality. So when f is continuous at x the Fourier series converges to $f(x_0)$, and when it is discontinuous it converges to the mid-point of the jump $\frac{1}{2}[f(x_{0-} - t) + f(x_{0+} + t)]$.

Exercise Set 9.3

1. $b_1 = \dfrac{2}{\pi}(\pi^2 - 4)$, $b_2 = -\pi$, $b_3 = \dfrac{2}{27\pi}(9\pi^2 - 4)$,
$b_4 = -\dfrac{\pi}{2}$, $b_5 = \dfrac{2}{125\pi}(25\pi^2 - 4)$

3. $b_1 = 1/\pi$, $b_2 = 4/(3\pi)$, $b_3 = 1/\pi$, $b_4 = 8/(15\pi)$, $b_5 = 1/(3\pi)$

5. $\dfrac{1}{\pi} + \dfrac{1}{2}\cos x + \dfrac{2}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\dfrac{\cos 2nx}{(4n^2-1)}$

7. $\dfrac{1}{\pi} + \dfrac{1}{\pi}\cos x - \dfrac{2}{3\pi}\cos 2x - \dfrac{1}{\pi}\cos 3x - \dfrac{2}{15\pi}\cos 4x + \dfrac{1}{3\pi}\cos 5x - \dfrac{2}{35\pi}\cos 6x + \cdots$

11. $\dfrac{2}{\pi}\sum_{n=1}^{\infty}[1 + (-1)^{n+1}e^{-\pi}]\dfrac{n\sin nx}{(n^2+1)}$

13. The linearity of the integral used in the derivation of the Fourier series coefficients allows the Fourier series of $f(x) \pm g(x)$ to be added or subtracted term by term. The Parseval relation gives

$$\frac{1}{\pi}\int_{-\pi}^{\pi}[f(x) \pm g(x)]^2 dx$$
$$= a_0 \pm A_0 + \sum_{n=1}^{\infty}[(a_n \pm A_n)^2 + (b_n \pm B_n)^2].$$

The result follows by subtracting the result with the negative sign from the corresponding result with the positive sign.

Exercise Set 9.4

1. $\frac{1}{2} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{(2n-1)}$

3. $\frac{\pi}{2} - \sum_{n=1}^{\infty}\frac{\sin n\pi x}{n}$

5. $c_n = (-1)^n \frac{\sinh 1(1 - in\pi)}{1 + n^2\pi^2}, \quad n = 0, \pm 1, \pm 2, \ldots$

7. $c_n = \frac{e-1}{1 - 2n\pi i}, \quad n = 0, \pm 1, \pm 2, \ldots$

9. $c_n = (-1)^n \frac{\sinh \pi}{\pi(1 - in)}, \quad n = 0, \pm 1, \pm 2, \ldots$

Exercise Set 9.5

1. $\omega_0 = 1/2, \quad f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos\frac{1}{2}(2n-1)x}{(2n-1)^2}$

$$+ 2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin\frac{1}{2}nx}{n} \quad A_0 = \frac{\pi}{2},$$

$$A_1 = \left[\left(\frac{4}{\pi}\right)^2 + 2\right]^{1/2}, \quad A_2 = 1,$$

$$A_3 = \left[\left(\frac{4}{5\pi}\right)^2 + \left(\frac{2}{3}\right)^2\right]^{1/2}, \quad A_4 = \frac{1}{2}, \ldots$$

3. $\omega_0 = 1, \quad f(x) = -2 - \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{(2n-1)},$

$$A_0 = 2, \quad A_{2n-1} = \frac{8}{\pi(2n-1)},$$

$$A_{2n} = 0, \quad n = 1, 2, \ldots$$

5. $\omega_0 = 4, \ f(x) = \frac{\pi^2}{48} + \frac{1}{4}\sum_{n=1}^{\infty}(-1)^n\frac{\cos 4nx}{n^2},$

$$A_0 = \frac{\pi^2}{48}, \quad A_n = \frac{1}{4n^2}, \quad n = 1, 2, \ldots$$

Exercise Set 9.6

3. Case (d); $\quad d_{mn} = (-1)^{m+n}\frac{4}{m^3 n}[m^2\pi^2 - 6]$

5. Case (d); $\quad d_{mn} = \frac{16}{mn\pi}$ for m, n odd and $d_{mn} = 0$ for m, n even

7. Case (d); $\quad d_{mn} = (-1)^{m+n}\frac{32}{\pi^2 mn}$

9. Case (d); $\quad (-1)^{m+1}\frac{4}{mn^3\pi}\{2[(-1)^n - 1] + (-1)^{n+1}n^2\pi^2\}$

Exercise Set 10.1

1. $A(\omega) = \frac{2\sin\omega}{\omega\pi}, \ B(\omega) \equiv 0,$

$$f(x) = \frac{2}{\pi}\int_0^{\infty}\frac{\cos\omega x \sin\omega}{\omega}d\omega$$

3. $A(\omega) \equiv 0, \ B(\omega) = \frac{2b}{\omega^2 a\pi}(\sin\omega a - \omega a\cos\omega a),$

$$f(x) = \frac{2b}{a\omega}\int_0^{\infty}\frac{\sin\omega x(\sin\omega a - \omega a\cos\omega a)}{\omega^2}d\omega$$

When $x = a$, $\frac{1}{2}[f(a+0) + f(a-0)] = b/2$, so this result also shows that

$$\int_0^{\infty}\frac{\sin\omega a(\sin\omega a - \omega a\cos\omega a)}{\omega^2}d\omega = \frac{\pi a}{4}$$

5. $f(x) = \int_0^{\infty}\frac{\cos\frac{1}{2}\omega\pi\cos\omega x}{1 - \omega^2}d\omega$

7. $f(x) = \frac{1}{\pi}\int_0^{\infty}\frac{\omega[\sin\omega x - \sin\omega(\pi + x)]}{\omega^2 - 1}d\omega$

Exercise Set 10.3

11. $F_C\{f(x)\} = \sqrt{\frac{2}{\pi}}\left(\frac{1 + \cos\omega\pi}{1 - \omega^2}\right)$

13. $F_C\{f(x)\} = \sqrt{\frac{2}{\pi}}\left(\frac{2\cos\omega - 1 - \cos 2\omega}{\omega^2}\right)$

15. $F_C\{f(x)\} = 2\sqrt{\frac{2}{\pi}}\left(\frac{\sin\omega - \omega\cos\omega}{\omega^3}\right)$

25. $F_S\{f(x)\} = -\sqrt{\frac{2}{\pi}}\left(\frac{\omega(1 + \cos\omega\pi)}{1 - \omega^2}\right)$

27. $F_S\{f(x)\} = \sqrt{\frac{2}{\pi}}\left(\frac{\omega - \sin 2\omega + \sin\omega}{\omega^2}\right)$

Exercise Set 11.1

1. $d\mathbf{r}/dt = (\sin t + t\cos t)\mathbf{i} + (\cos t - t\sin t)\mathbf{j} + 2t\mathbf{k}$, $(d\mathbf{r}/dt)_{t=\pi/2} = \mathbf{i} - (\pi/2)\mathbf{j} + \pi\mathbf{k}$

$d^2\mathbf{r}/dt^2 = (2\cos t - t\sin t)\mathbf{i} - (2\sin t + t\cos t)\mathbf{j} + 2\mathbf{k}$, $(d^2\mathbf{r}/dt^2)_{t=\pi/2} = -(\pi/2)\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$

3. $d\mathbf{r}/dt = 2\sin t\cos t\,\mathbf{i} + 2\sin t\cos t\,\mathbf{j} - \mathbf{k}$, $(d\mathbf{r}/dt)_{t=\pi/4} = \mathbf{i} + \mathbf{j} - \mathbf{k}$
$d^2\mathbf{r}/dt^2 = 2(\cos^2 t - \sin^2 t)\mathbf{i} + 2(\cos^2 t - \sin^2 t)\mathbf{j}$, $(d^2\mathbf{r}/dt^2)_{t=\pi/4} = \mathbf{0}$

5. $d\mathbf{r}/dt = (1-\cos t)\mathbf{i} + \sin t\,\mathbf{j}$, $(d\mathbf{r}/dt)_{t=\pi/2} = \mathbf{i} + \mathbf{j}$
$d^2\mathbf{r}/dt^2 = \sin t\,\mathbf{i} + \cos t\,\mathbf{j}$, $(d^2\mathbf{r}/dt^2)_{t=\pi/2} = \mathbf{i}$

9. $d\mathbf{r}/ds = 2s\mathbf{i}/(1+s^2) + 12s\ln(1+s^2)\mathbf{j}/(1+s^2) - 2s\mathbf{k}/(1+s^2)$

11. $d\mathbf{r}/dt = 2t\mathbf{i} - 8\sin 2t\,\mathbf{j} + 6\cos 2t\,\mathbf{k}$. A unit vector in the given direction is $\hat{\mathbf{a}} = \frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$ so the component in the required direction is $\hat{\mathbf{a}}\cdot d\mathbf{r}/dt = \frac{4}{3}t - \frac{8}{3}\sin 2t + 4\cos 2t$

19. $\dfrac{d}{dt}\{\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})\} = -4t^3 - 36t^2 - 6t + 4$

21. $\mathbf{T} = \dfrac{1}{(a^2\omega^2 + b^2)^{1/2}}[-a\omega\sin\omega t\,\mathbf{i} + a\omega\cos\omega t\,\mathbf{j} + b\mathbf{k}]$

$\mathbf{N} = -\cos\omega t\,\mathbf{i} - \sin\omega t\,\mathbf{j}$

$\mathbf{B} = \dfrac{1}{(a^2\omega^2 + b^2)^{1/2}}[b\sin\omega t\,\mathbf{i} - b\cos\omega t\,\mathbf{j} + a\omega\mathbf{k}]$

$\kappa = \dfrac{a\omega^2}{(a^2\omega^2 + b^2)}$

Exercise Set 11.2

1. (a) $((1/4)\sin 2t - (1/2)t\cos 2t)\mathbf{i} + t^3\mathbf{j} - (3/2)t^2\mathbf{k}$
(b) $[(7/3)\ln 7 - 2]\mathbf{i} + (1 + e^2)\mathbf{k}$

3. (a) $[(1/6)\cos 3t\sin 3t + t(1/2)]\mathbf{i} + (1/2)[t - \cos t\sin t]\mathbf{j} + (1/2)t^2\mathbf{k}$
(b) $(\pi + \pi^3)\mathbf{i} + (1/3)\mathbf{k}$ **5.** $(\pi/2)(a^2 + \alpha^2)^{1/2}$

7. Integrate $\mathbf{F}\cdot d\mathbf{r}$ between the limits $t = 0$ and $t = \pi/2$ to obtain $\pi/4$

9. $2\pi^2$ **10.** 4 **11.** (a) 0, (b) $-3\pi/4$ **13.** 8π

Exercise Set 11.3

1. $\sqrt{5}(\pi + 2\sqrt{2})/10$ **3.** $(15e^{-2} - 2)/\sqrt{17}$

5. $\sqrt{2}[(\pi/8) - 1]/3 + 2e^3$

7. $4\sqrt{5}\cosh 2$

11. $(2x + 3yz)\mathbf{i} + (3xz - z^2)\mathbf{j} + (3xy - 2yz)\mathbf{k}$

13. $[(y - 3z)\mathbf{i} + (x + 2z)\mathbf{j} + (2y - 3x)\mathbf{k}]\exp(xy + 2yz - 3xz)$

15. A normal $\mathbf{n}$ to $f(x, y) =$ constant is $\mathbf{n} = \text{grad } f$, so at point $P(x_0, y_0)$, $\mathbf{n} = (\text{grad } f)_P$, so $\mathbf{n} = (f_x)_P\mathbf{i} + (f_y)_P\mathbf{j}$. The vector equation of a line normal to f at P is $\mathbf{r} = \mathbf{r}_0 + \lambda(\text{grad } f)_P$ with $\mathbf{r}_0 = x_0\mathbf{i} + y_0\mathbf{j}$. The cartesian equation is found by eliminating λ between $x = x_0 + \lambda(f_x)_P$ and $y = y_0 + \lambda(f_y)_P$ to obtain $y = y_0 + (x - x_0)(f_x/f_y)_P$.

17. A normal to the surface is grad f, so at $(1, 2, 2)$ the normal $\mathbf{n} = 9\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$. The tangent plane through $\mathbf{r}_0 = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ is $(\mathbf{r} - \mathbf{r}_0)\cdot\mathbf{n} = 0$, so the plane has the equation $9x + 3y + 4z = 23$.

19. The normal to the surface at $\mathbf{r}_0$ is $(\text{grad } f)_{\mathbf{r}_0}$ so the required equation is $(\mathbf{r} - \mathbf{r}_0)\cdot(\text{grad } f)_{\mathbf{r}_0} = 0$.

21. $(2r\sin\theta + z^2)\mathbf{e}_r + r\cos\theta\,\mathbf{e}_\theta + 2rz\mathbf{e}_z$

23. $\text{grad}(f^n) = nf^{n-1}(f_x\mathbf{i} + f_y\mathbf{j} + f_z\mathbf{k}) = nf^{n-1}\mathbf{F}$
If $f = r$ then $f = (x^2 + y^2 + z^2)^{1/2}$ and $\text{grad } r = (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})/(x^2 + y^2 + z^2)^{1/2} = \hat{\mathbf{r}}$.
If $f = 1/r$ then $\text{grad } f = -(1/r^2)\text{grad } r = -(1/r^2)\hat{\mathbf{r}} = -\mathbf{r}/r^3$.

Exercise Set 11.4

1. Yes **3.** No **5.** No

7. $f = xz^3 + 3x^2y^2 + \text{constant}$; $I = f(Q) - f(P) = 11$

9. $f = x\exp(xyz) + \text{constant}$; $I = f(Q) - f(P) = e^2$

11. $f = x^2 + x^2yz^2 + \text{constant}$; $I = f(Q) - f(P) = -17$

Exercise Set 11.5

1. $\text{div }\mathbf{F} = 2xy + 2yz^2 + 3xz^2$

3. $\text{div }\mathbf{F} = 6x + 4x^2y$

5. Substitute $\phi\mathbf{F}$ into the definition of divergence and expand the result.

7. $\text{curl }\mathbf{F} = (2xy - x^2y)\mathbf{i} + (2xyz - y^2)\mathbf{j} + (2xyz - xz^2)\mathbf{k}$

9. $\text{curl }\mathbf{F} = \mathbf{i} + \dfrac{x(3y^2 + 2x^2)}{(x^2 + 2y^2)(x^2 + y^2)}\mathbf{k}$

11. Expand curl $\mathbf{F}$, substitute into the definition of divergence, and make use of the equality of mixed derivatives.

13. Substitute $\mathbf{F}\cdot\mathbf{G}$ into the definition of grad and expand the result.

15. Substitute $\mathbf{F}\times\mathbf{G}$ into the definition of curl and expand the result.

17. $\nabla^2\mathbf{F} = \mathbf{0}$, so $\text{curl}(\text{curl}\mathbf{F}) = \text{grad div }\mathbf{F} - \nabla^2\mathbf{F} = \text{grad div }\mathbf{F} = 3(z\mathbf{i} + y\mathbf{k})$

21. Yes; $f = \ln(1 + x^2 + 2y^2z) = \text{constant}$

Exercise Set 11.6

1. $\nabla \cdot (a\mathbf{F}) = a\nabla \cdot \mathbf{F}$; $\nabla \cdot (a\mathbf{F} + b\mathbf{G}) = a\nabla \cdot \mathbf{F} + b\nabla \cdot \mathbf{G}$;
$\nabla \cdot (\phi\mathbf{F}) = \phi\nabla \cdot \mathbf{G} + \mathbf{F} \cdot \nabla\phi$;
$\nabla \cdot (\nabla\phi) = \nabla^2\phi$; $\nabla \cdot (\phi\nabla\psi) = \phi\nabla^2\psi + \nabla\phi \cdot \nabla\psi$;
$\nabla \cdot (\phi\nabla\psi) - \nabla \cdot (\psi\nabla\phi) = \phi\nabla^2\psi - \psi\nabla^2\phi$

5. $h_1 = h_2 = \sqrt{2}$, $h_3 = \cosh q_3$; $\mathbf{q} = (q_1 - q_2)\mathbf{i} + (q_1 + q_2)\mathbf{j} + \sinh q_3\mathbf{k}$; $\mathbf{e}_1 = \dfrac{1}{h_1}\dfrac{\partial \mathbf{q}}{\partial q_1} = \dfrac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j})$, $\mathbf{e}_2 = \dfrac{1}{h_2}\dfrac{\partial \mathbf{q}}{\partial q_2} = \dfrac{1}{\sqrt{2}}(-\mathbf{i} + \mathbf{j})$, $\mathbf{e}_3 = \mathbf{k}$, so $\mathbf{e}_1, \mathbf{e}_2$, and $\mathbf{e}_3$ form an orthonormal set.

$$\text{grad } f = \mathbf{e}_1\frac{1}{\sqrt{2}}\frac{\partial f}{\partial q_1} + \mathbf{e}_2\frac{1}{\sqrt{2}}\frac{\partial f}{\partial q_2} + \mathbf{e}_3\frac{1}{\cosh q_3}\frac{\partial f}{\partial q_3}$$

$$\text{div } \mathbf{F} = \frac{1}{\sqrt{2}}\frac{\partial F_1}{\partial q_1} + \frac{1}{\sqrt{2}}\frac{\partial F_2}{\partial q_2} + \frac{1}{\cosh q_3}\frac{\partial F_3}{\partial q_3}$$

7. $h_1 = h_2 = \sinh^2\xi + \sin^2\eta$, $h_3 = 1$
$\mathbf{q} = \cosh\xi\cos\eta\mathbf{i} + \sinh\xi\sin\eta\mathbf{j} + z\mathbf{k}$

$\mathbf{e}_\xi = \dfrac{1}{h_1}\dfrac{\partial \mathbf{q}}{\partial \xi} = \dfrac{1}{\sinh^2\xi + \sin^2\eta}(\sinh\xi\cos\eta\mathbf{i} + \cosh\xi\sin\eta\mathbf{j})$

$\mathbf{e}_\eta = \dfrac{1}{h_1}\dfrac{\partial \mathbf{q}}{\partial \eta} = \dfrac{1}{\sinh^2\xi + \sin^2\eta}(-\cosh\xi\sin\eta\mathbf{i} + \sinh\xi\cos\eta\mathbf{j})$

$\mathbf{e}_z = \mathbf{k}$, so $\mathbf{e}_\xi$, $\mathbf{e}_\eta$, and $\mathbf{e}_z$ form an orthonormal set. $\xi =$ constant are ellipses and $\eta =$ constant are hyperbolas

$$\text{grad } f = \frac{1}{\sinh^2\xi + \sin^2\eta}\frac{\partial f}{\partial \xi}\mathbf{e}_\xi + \frac{1}{\sinh^2\xi + \sin^2\eta} \times \frac{\partial f}{\partial \eta}\mathbf{e}_\eta + \mathbf{e}_z\frac{\partial f}{\partial z}$$

Exercise Set 12.2

1. Set $\mathbf{F} = \mathbf{a} \times \mathbf{G}$ in the divergence theorem to obtain

$$\iint_S (\mathbf{a} \times \mathbf{G}) \cdot d\mathbf{S} = \iiint_D \text{div}(\mathbf{a} \times \mathbf{G})dV \text{ but}$$
$$\text{div}(\mathbf{a} \times \mathbf{G}) = -\mathbf{a} \cdot \text{curl } \mathbf{G}, \text{ so}$$
$$\iint_S (\mathbf{a} \times \mathbf{G}) \cdot d\mathbf{S} = -\iiint_D \mathbf{a} \cdot \text{curl}\mathbf{G}dV \text{ or}$$
$$\iint_S (\mathbf{a} \times \mathbf{G}).\mathbf{n}dS = -\iiint_D \mathbf{a} \cdot \text{curl}\mathbf{G}dV$$

The properties of the scalar triple product allow the interchange of the dot and the cross to give (because $\mathbf{a}$ is a constant vector)

$\mathbf{a} \cdot \displaystyle\iint_S \mathbf{G} \times d\mathbf{S} = -\mathbf{a} \cdot \iiint_D \text{curl}\mathbf{G}dV$. As $\mathbf{a}$ is arbitrary this last result implies that $\displaystyle\iint_S \mathbf{G} \times d\mathbf{S} = -\iiint_D \text{curl}\mathbf{G}dV$.

3. Set $\mathbf{F} = \phi\mathbf{G}$ in the divergence theorem and use the result that $\text{div}(\phi\mathbf{G}) = (\text{grad }\phi) \cdot \mathbf{G} + \phi \text{ div } \mathbf{G}$

5. Write $\text{div}(\kappa T\text{grad } T) = \text{div}(T[\kappa\text{grad } T])$ and expand the expression to get $\text{div}(\kappa T\text{grad } T) = (\text{grad } T) \cdot (\kappa\text{grad } T) + T \text{ div } (\kappa\text{grad } T)$, so the heat equation becomes $\text{div}(\kappa T \text{ grad } T) = \kappa(\text{grad } T) \cdot (\text{grad } T) + \mu\rho T\partial T/\partial t$. Now integrate over D and use the divergence theorem to get

$$\iint_S \kappa T(\text{grad } T) \cdot d\mathbf{S} = \iiint_D \kappa(\text{grad } T) \cdot (\text{grad } T)dV + \iiint_D \mu\rho T\frac{\partial T}{\partial t}dV$$

7. Replace $\mathbf{F}$ in Stokes's theorem by ϕ $\mathbf{F}$ and use curl $(\phi\mathbf{F}) = (\text{grad }\phi) \times \mathbf{F} + \phi \text{ curl } \mathbf{F}$

Exercise Set 12.3

1. Reason as in Example 12.16 with $\mathbf{q} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$

3.
$$\frac{d}{dt}\iiint_{D(t)} f(\mathbf{r}, t)dV = \frac{d}{dt}\left[\int_0^1\int_0^1\int_{ut}^{vt} xytdzdydx\right] = \frac{d}{dt}\left[\frac{1}{4}(v-u)t^2\right] = \frac{1}{2}(v-u)t$$

Here, on the upper surface $\mathbf{q} = v\mathbf{k}$ so $d\mathbf{S} = dxdy\mathbf{k}$, while on the lower surface $\mathbf{q} = u\mathbf{k}$ and $d\mathbf{S} = -dxdy\mathbf{k}$, so

$$\iiint_{D(t)} \frac{\partial f(\mathbf{r}, t)}{\partial t}dV + \iint_{S(t)} f\mathbf{q} \cdot d\mathbf{S} = \int_0^1\int_0^1\int_{ut}^{vt} xydxdydz + \int_0^1\int_0^1 xytvdydx - \int_0^1\int_0^1 xytudydx = \frac{1}{2}(v-u)t,$$

so the two results are in agreement.

5. Use cylindrical symmetry when evaluating the integrals with $dV = 2\pi rhdr$ and $d\mathbf{S} = hr\,d\theta$.

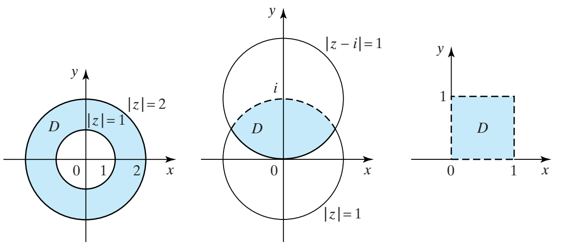
$$\frac{d}{dt}\iiint_{D(t)} f(\mathbf{r}, t)dV = \frac{d}{dt}\left[\int_0^{ut} r^2 t 2\pi rhdr\right] = \frac{5}{2}\pi hu^4t^4 \quad \text{and}$$

$$\iiint_{D(t)} \frac{\partial f(\mathbf{r}, t)}{\partial t}dV + \int_{S(t)} f\mathbf{q}\cdot d\mathbf{S} = \int_0^{ut} r^2 2\pi rhdr + hu^4t^4\int_0^{2\pi} d\theta = \frac{5}{2}\pi hu^4t^4,$$

so the two results are in agreement.

Exercise Set 13.1

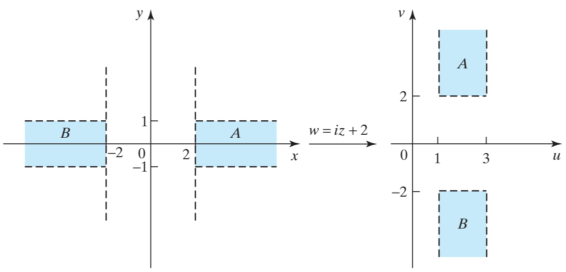
1.



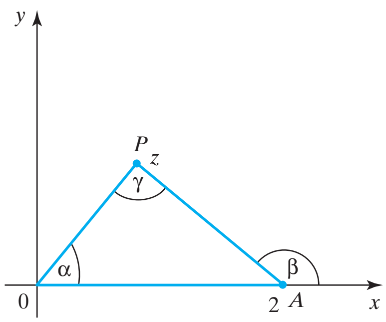
(a) Closed set (b) Region (c) Open set

3. line $y = -x$ from the origin to the point $(-2, -2)$

5.



7.



Angle $OAP = \pi - \beta$, but $\alpha +$ angle $OAP + \gamma = \pi$, so $\gamma = \beta - \alpha$. As $\alpha = \text{Arg}\, z$, $\beta = \text{Arg}\,(z-2)$, so $\text{Arg}\,(z-2) - \text{Arg}\, z = \gamma = \pi/2$. From Euclidean geometry point P must lie on a circle with its diameter from the point (0, 0) to (2, 0). The condition $0 \le \text{Arg}\, z \le \pi/2$ defines the part of the circle that lies in the upper half of the z-plane.

9. An ellipse with the foci at $z = \pm 1$ and eccentricity $e = 1/2$

11. $f(z)$

$$= \left(\frac{2x^2 + 2y^2 + 3y + 1}{x^2 + (1+y)^2}\right) - i\left(\frac{x}{x^2 + (1+y)^2}\right)$$

$$= \left(\frac{2r^2 + 3r\sin\theta + 1}{r^2 + 2r\sin\theta + 1}\right) - i\left(\frac{r\cos\theta}{r^2 + 2r\sin\theta + 1}\right)(z \ne 0)$$

$u = \text{Re}\{f(z)\}$, $v = \text{Im}\{f(z)\}$

13.

$$f(z) = e^{-y}(x\cos x - y\sin x) + ie^{-y}(y\cos x + x\sin x)$$
$$= r\exp(-r\sin\theta)\{\cos\theta\cos(r\cos\theta) - \sin\theta\sin(r\cos\theta)\} + ir\exp(-r\sin\theta)\{\sin\theta\cos(r\cos\theta) + \cos\theta\sin(r\cos\theta)\}$$

$u = \text{Re}\{f(z)\}$, $v = \text{Im}\{f(z)\}$

Exercise Set 13.2

1. $\text{Re}\{f(x)\} = x^3 - 3xy^2 + 4x^2 - 4y^2 - 3x + 1$, $\text{Im}\{f(x)\} = 3x^2y - y^3 + 8xy - 3y$; continuous for all z

3.
$$\text{Re}\{f(z)\} = \frac{2xy^2 + x(1 + x^2 - y^2)}{(1 + x^2 - y^2)^2 + 4x^2y^2},$$
$$\text{Im}\{f(z)\} = \frac{y(1 + x^2 - y^2) - 2x^2y}{(1 + x^2 - y^2)^2 + 4x^2y^2};$$

discontinuous at $z = \pm i$

5. $f'(z) = 3z^2 + 1$ for all z

7. $f'(z) = -1/(1+z)^2$ for $z \ne -1$

9. $f'(z) = 3z^2$ for all z

11. $f'(z) = 1 - 1/z^2$ for $z \ne 0$

13. Substitute in the definitions of the functions on the right and show they simplify to the function on the left. The second result follows by setting $z_1 = x$ and $z_2 = iy$ and using $\cosh(iy) = \cos y$ and $\sinh(iy) = i\sin y$.

15. To establish the first identity substitute in the definitions of the functions on the left and show they

simplify to unity. The second identity follows from the first one after division by $\cosh^2 z$ and rearrangement of the result.

17. In the first identity substitute in the definitions of the functions on the right and show they simplify to the function on the left. The second result follows from the first by setting $z_1 = x$ and $z_2 = iy$ and using $\cos(iy) = \cosh y$ and $\sin(iy) = i \sinh y$.

19. Establish the first identity by substituting into the definitions of the functions on the left and showing the result simplifies to unity. The second result follows from the first after division by $\cos^2 z$.

21. $z = n\pi, n = 0, \pm 1, \pm 2, \ldots$

23. $z = n\pi i, n = 0, \pm 1, \pm 2, \ldots$

25. $z = (2n+1)\pi \pm 3i, n = 0, \pm 1, \pm 2, \ldots$

27. $z = \pm 2 + (4n+1)\pi i/2, n = 0, \pm 1, \pm 2, \ldots$

29. $z = n\pi i, n = 0, \pm 1, \pm 2, \ldots$ (the zeros of $\sinh z$)

31. (a) $0, \pm\pi, \sqrt{3}e^{i\pi/4}, \sqrt{3}e^{5i\pi/4}$ (b) $z = 2\{\cos(2k+1)\pi/4 + i \sin(2k+1)\pi/4\}, k = 0, 1, \ldots$ (c) Nowhere analytic because $|z|$ is not an analytic function

33. $3 \cos 3x \cosh 3y - i3 \sin 3x \sinh 3y = 3 \cos 3z$

35. Using the change of variables from cartesian to polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, substitute in the change of variable formulas $u_x = r_x u_r + \theta_x u_\theta$ etc. to find u_x, u_y, v_x and v_y. Use these results in the cartesian form of the Cauchy–Riemann equations to obtain their polar form.

37. $f'(z) = 1 - 1/z^2$

39. $f(z) = 3z^3 + z + 1,\ f'(z) = 9z^2 + 1$

Exercise Set 13.3

1. $f(z) = z^3 + (2-i)z + ic$

3. $f(z) = ze^{iz} + 2iz + a$

5. $f(z) = z \sinh 2z + a$ **7.** $f(z) = z \cos 3z + ic$

9. $f(z) = z + (2-i)z^2 + ic$

11. Show that the functions do not satisfy the Cauchy–Riemann equations.

13. Say $u \equiv$ constant. Then from the Cauchy–Riemann equations $v_x = v_y = 0$, so $v =$ constant, and hence $f(z) = u + iv \equiv$ constant in D. If $f(z)$ is not analytic there is no connection between u and v, so if $u \equiv 0$ it is not necessary that $v = 0$. A simple example is $f(z) = |z| + i$ constant.

15. Combine similar terms and chose a and b to make $\Delta\Phi = 0$ to get $a = 1, b = -2$.

Exercise Set 13.4

1. $(4n+1)\pi/2 - i\ln(\sqrt{5}+2)$ using the principal value of the square root function. $(4n-1)\pi/2 - i\ln(\sqrt{5}-2)$ using the value from the second branch of the square root function. $\pi/2 - i\ln(\sqrt{5}+2)$ using the principal values of the square root and logarithmic functions.

3. $(4n+1)\pi i/4, \pi i/4$ using the principal value of the logarithmic function.

5. $-(1/8)(8n+1)\pi + (1/4)i\ln 2,\ \ -\pi/8 + (1/4)i\ln 2$ using the principal branch of the logarithmic function.

7. $\arcsin z + \arccos z = -i\log[iz + (1-z^2)^{1/2}] - i\log[z + i(1-z^2)^{1/2}] = -i\log\{[iz + (1-z^2)^{1/2}][z + i(1-z^2)^{1/2}]\} = -i\log i$. However, as $i = e^{i\pi/2} \cdot e^{2n\pi i}$, so $-i\log i = \pi/2 + 2n\pi$.

9. From (59) $\log z = \ln|z| + i \operatorname{Arg} z$ so immediately above the negative real axis $\operatorname{Arg} a = \pi$ and immediately below it $\operatorname{Arg} z = -\pi$, so there is a jump of $2\pi i$ across the negative real axis.

Exercise Set 14.1

1. $AB: z = t + it/2, 2 \le t \le 4$
$BC: z = t + i(2t-6), 4 \le t \le 5$

3. $AB: z = t + i(2t-5), 3 \le t \le 4$
$BC: z = 4 - t + i(3+t), 0 \le t \le 3$

5. 0 **7.** $-18 - 18i$

9. $36 + 21i$ **11.** $\cosh 3 - \cosh 6$

13. $\cosh\pi(\cos 2 - \cos 3) + i\sinh\pi(\sin 3 - \sin 2)$

15. $(1/2)(\sinh 8 \cos 4 + i \cosh 8 \sin 4)$

17. $e^4/\sqrt{2} - 1 + ie^4/\sqrt{2}$

19. On the semicircle $\Gamma: z = 1 + e^{it}$, from $t = \pi$ to $t = 0$ (in the negative sense)

$$\int_\Gamma \frac{dz}{z-1} = \int_\pi^0 \frac{1}{e^{it}} ie^{it}\,dt = -\pi i$$

21. $\Gamma: z = 2 + 2e^{it}$ and as integration is in the positive sense $\displaystyle\int_\Gamma \frac{1}{z+i}dz = \int_0^{2\pi} \frac{1}{2+2e^{it}+i} 2ie^{it}\,dt = [\log(2+2e^{it}+i]_0^{2\pi} = 0$. Reversal of the direction of integration gives the same result.

Exercise Set 14.2

1. $\cos 1 - (1/2)(e + 1/e)$; $f(z)$ is analytic, so Theorem 14.4 applies.

3. $5/2+3i$; $f(z)$ is not analytic, so Theorem 14.4 cannot be used.

5. 0; $f(z)$ is analytic in $|z|\le 1$, so the Cauchy–Goursat theorem applies.

7. $0; z$ is analytic but $\bar{z}^2$ is not, so $\int_\Gamma f(z)dz = \int_\Gamma z dz + \int_\Gamma \bar{z}^2 dz = 0+0=0$.

9. (a) The points $\pm i$ must not lie inside Γ. (b) The points $z=n\pi, n=0,\pm 1,\ldots$ (the zeros of $\sin z$) must not lie inside Γ. (c) The points $z=(2n+1)i\pi/2, n=0,\pm 1\ldots$ (the zeros of $\cosh z$) must not lie inside Γ. (d) The points $z=n\pi i, n=0,\pm 1,\ldots$ must not lie inside Γ.

11. $f(z)=\dfrac{z+5}{z^2+3z-4}=\dfrac{6}{5}\dfrac{1}{z-1}-\dfrac{1}{5}\dfrac{1}{z+4}$ so

(a) $\displaystyle\int_\Gamma f(z)dz=\frac{6}{5}\int_\Gamma\frac{dz}{z-1}+0=\frac{12\pi i}{5}$

(b) $\displaystyle\int_\Gamma f(z)dz=0-\frac{1}{5}\int_\Gamma\frac{dz}{z+1}+0=-\frac{2\pi i}{5}$

13. $f(z)=\dfrac{2-7z}{z^2+3z}=\dfrac{2}{3}\dfrac{1}{z}-\dfrac{23}{3}\dfrac{1}{z+3}$ so

(a) $\displaystyle\int_\Gamma f(z)dz=\frac{2}{3}\int_\Gamma\frac{dz}{z}+0=\frac{4\pi i}{3}$

(b) $\displaystyle\int_\Gamma f(z)dz=0-\frac{23}{3}\int_\Gamma\frac{dz}{z+3}=-\frac{46\pi i}{3}$

15. $f(z)=\dfrac{z^2+2z}{z^2-2z+1}=1+\dfrac{3}{(z-1)^2}+\dfrac{4}{z-1}$;

$$\int_\Gamma f(z)dz=0+0+4\int_\Gamma\frac{dz}{z-1}=8\pi i$$

17. $f(z)=\dfrac{2z-1}{(z+1)^3}=\dfrac{2}{(z+1)^2}-\dfrac{3}{(z+1)^3}$;

$$\int_\Gamma f(z)dz=2\int_\Gamma\frac{dz}{(z+1)^2}-3\int_\Gamma\frac{dz}{(z+1)^3}$$
$$=0-0=0$$

Exercise Set 14.3

1. 0

3. $\pi i/\sqrt{2}$

5. $2\pi e^4 i$

7. $\pi \sin 1$

9. $\pi i\sqrt{2}\left(\dfrac{\pi}{86}-\dfrac{1}{6}\right)$

11. $-2\pi i$

13. $\dfrac{\pi i}{3}(5\cos 1-6\sin 1)$

15. $-\dfrac{\pi}{2}e^{-i}$

17. Set $z-z_0=Re^{i\theta}$ in the Cauchy integral formula for derivatives, take the absolute value, and use the integral inequality in Theorem 14.1 to obtain

$$|f^n(z_0)|\le\left|\frac{n!}{2\pi i}\int_0^{2\pi}\frac{f(z)Rie^{i\theta}}{R^{n+1}e^{i(n+1)\theta}}d\theta\right|$$
$$\le\frac{n!M}{2\pi R^n}\int_0^{2\pi}d\theta=\frac{n!M}{R^n}.$$

19. $$\int_\Gamma\frac{d}{dt}\left[\frac{(t^2-1)^{n+1}}{(t-z)^{n1}}\right]dt$$
$$=\int_\Gamma\frac{(n+1)(t^2-1)^n(t^2-2tz+1)}{(t-z)^{n+2}}dt=0.$$

Express $P'_{n+1}(z)-zP'_n(z)-(n+1)P_n(z)$ in terms of the integral definition of $P_n(z)$ to show that apart from a constant factor it is given by the contour integral in Exercise 18, so $P'_{n+1}(z)-zP'_n(z)-(n+1)P_n(z)=0$.

21. $$\int_\Gamma\frac{d}{dt}\left[\frac{t(t^2-1)}{(t-z)^n}\right]dt=\int_\Gamma\left[\frac{(t^2-1)^n}{(t-z)^n}\right.$$
$$\left.+2\frac{nt^2(t^2-1)^{n-1}}{(t-z)^n}-\frac{nt(t^2-1)^n}{(t-z)^{n+1}}\right]dt=0.$$

Express $(n+1)P_{n+1}(z)-(2n+1)zP_n(z)+nP_{n-1}(z)$ in terms of the integral definition of $P_n(z)$ to show that apart from a constant factor it is given by the contour integral in Exercise 18, so $(n+1)P_{n+1}(z)-(2n+1)zP_n(z)+nP_{n-1}(z)=0$.

23. Perform the indicated differentiation in Exercise 22 to obtain an equivalent expression for that result. Construct $G(z)$ using the integral representation for $P_n(z)$ and show that after simplification it reduces to $G(z)$. As Exercise 22 establishes that $G(z)=0$ it follows that the Legendre differential equation is $(1-z^2)P''_n(z)-2zP'(z)+n(n+1)P_n(z)=0$.

Exercise Set 14.4

1. If $0\le k\le n$,

$$\frac{1}{2\pi i}\frac{P_k}{z^{k+1}}=\frac{1}{2\pi i}\left[\frac{a_0}{z^{k+1}}+\frac{a_1}{z^k}+\cdots+\frac{a_k}{z}+a_{k+1}\right.$$
$$\left.+\cdots a_n z^{n-k+1}\right].$$

Integrating around Γ shows that all integrals but that of a_k/z vanish, while $\dfrac{1}{2\pi i}\displaystyle\int_\Gamma\frac{a_k}{z}dz=a_k$, so

$$\frac{1}{2\pi i}\sum_{k=0}^{n}\int_\Gamma\frac{P_n(z)}{z^{k+1}}dz=\sum_{k=0}^{n}a_k.$$

3. In terms of the given substitutions

$$f(re^{i\theta}) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{(R^2-r^2)f(e^{i\psi}R)}{e^{i\psi}R(z\bar{z}-z\bar{z}_0-z_0\bar{z}+z_0\bar{z}_0)}$$

$ie^{i\psi}Rd\psi$, but $z\bar{z}=R^2, z_0\bar{z}_0=r^2, z\bar{z}_0+z_0\bar{z}= rR\cos(\psi-\theta)$, so

$$f(re^{i\theta}) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{(R^2-r^2)f(Re^{i\psi})}{R^2-2rR\cos(\psi-\theta)+r^2}d\psi.$$

The Poisson integral formula follows from this by writing $f(re^{i\theta}) = u(r,\theta)+iv(r,\theta)$ and equating the real parts.

5. If z_0 lies inside the semicircle, then $\bar{z}_0$ lies outside it, so from the Cauchy integral formula $f(z_0) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z-z_0}dz$ and $0 = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z-\bar{z}_0}dz$. Subtracting these results and combining the integrands gives

$$\begin{aligned} f(z_0) &= \frac{1}{2\pi i}\int_\Gamma \frac{f(z)(z_0-\bar{z}_0)}{(z-z_0)(z-\bar{z}_0)}dz \\ &= \frac{1}{2\pi i}\int_\Gamma \frac{f(z)2iy_0}{(z-z_0)(z-\bar{z}_0)}dz \quad \text{where} \end{aligned}$$

$z_0 = x_0 + iy_0$

On the real axis

$z = x$ so $(z-z_0)(z-\bar{z}_0) = x^2-2xx_0+x_0^2+y_0^2 = |x-z_0|^2$ so

$$\begin{aligned} f(z_0) &= \frac{1}{2\pi i}\int_{-R}^{R} \frac{f(x)2iy_0}{|x-z_0|^2}dx \\ &\quad + \frac{1}{2\pi i}\int_{C_R} \frac{f(z)2iy_0}{(z-z_0)(z-\bar{z}_0)}dz, \end{aligned}$$

which after cancellation of the factors i and removal of the constant y_0 from the integrand gives the required result.

7. $P_n(z) = a_nz^n\left(1+\frac{a_{n-1}}{a_nz}+\frac{a_{n-2}}{a_nz^2}+\cdots+\frac{a_0}{a_nz^n}\right)$, so as $|z|\to\infty$ the bracketed term tends to 1, showing that $|P_n(z)|\to|a_nz^n|$ as $|z|\to\infty$. Thus, as $|z|\to\infty$, $|Q_n(z)|\to 1/|a_nz^n| = 1/(|a_n|r^n)$, showing that $|Q_n(z)|\to 0$ as $|z|\to\infty$.

9. $f(z) = e^z = e^{x+iy} = e^x(\cos y + i\sin y)$, so $|e^z| = e^x$. In $-1\le x\le 1, -2\le y\le 2, |e^z| = e^x$ has its greatest value e on $x=1$ for all y and its least value $1/e$ on $x=-1$ for all y, and thus $1/e < |e^z| < e$ for $-1\le x\le 1, -2\le y\le 2$.

11. $u = x+2x^2-2y^2$ is harmonic so the max/min of u occur on the boundary of the domain. Examination of u on the boundary shows Max $u=3$ at $x = 1, y=0$, and Min $u = -17/8$ at $x = -1/4, y = \pm 1$, so $-17/8 < u < 3$ inside the domain.

13. $u = e^x(x\cos y - y\sin y)$ is harmonic so the max/min of u occur on the boundary of the domain. Examination of u on the boundary shows Max $u = e$ at $x=1$ on $y=0$ and Min $u = -e\pi/2$ at $x = 1, y=\pm\pi/2$, so $-e\pi/2 < u < e$ in the domain.

Exercise Set 15.1

1. (a) Only cluster point is at 1, so the sequence converges to the limit 1, but the limit is not a member of the series.
(b) Cluster points at 0 and 4. The point 0 belongs to the sequence but the point 4 does not. The sequence has no limit.
(c) Only cluster point is at 5/2, so the sequence converges to the limit 5/2, but the limit is not a member of the sequence.

3. (a) This is one definition of the Euler number e, so the sequence converges to the limit e, but the limit is not a member of the sequence.
(b) Only cluster point is at $\pi/2$, so the sequence converges to $\pi/2$, but the limit is not a member of the sequence.
(c) Every member of the sequence is 1, so the sequence converges to the limit 1 that is a member of the sequence.

5. Convergent by comparison with $\Sigma 1/n^2$.

7. Divergent by comparison with $\Sigma 1/n$.

9. Divergent by nth root test as $L=2$.

11. Absolutely convergent by comparison with $\Sigma 1/n^2$ because for large n $\sin(1/n^2)\approx 1/n^2$.

13. Write $\frac{1}{r(r+1)} = \frac{1}{r}-\frac{1}{r+1}$ so

$$\begin{aligned} \sum_{r=1}^{n}\frac{1}{r(r+1)} &= \left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{2}{2}-\frac{1}{3}\right)+\cdots \\ &\quad + \left(\frac{1}{n}-\frac{1}{n+1}\right) = 1-\frac{1}{n+1}. \end{aligned}$$

So in the limit at $n\to\infty$ the series converges to 1. This cancellation of terms is called the *telescoping* of the series.

15. Convergent by nth root test because $L = (1/3)|2i-1|\lim\sqrt[n]{n} = \sqrt{5}/2 > 1$.

17. Use the approach in Exercise 13 to show that the series converges to 1.

19. Absolutely convergent by the nth root test.

21. $R = 2$; convergence for $|z| < 2$.

23. Alternate powers are missing so set $u = z^2$ and write as $2z\Sigma 2^n u^n/(4n+1)^2$. This has a radius of convergence $R = 1/2$, and so it converges for $|u| < 1/2$, and so for $|z| \ll 1/\sqrt{2}$.

25. $R = 0$; convergence only for $z = 0$.

27. $R = 2$; convergence for $|z| < 2$.

29. $R = 1$; convergence for $|z + 3| < 1$.

30. $R = 2$; convergence for $|z - 2| < 2$.

31. $R = 1$; convergence for $|z| < 1$.

33. $R = 1/2$; convergence for $|z| < 1/2$.

35. $\dfrac{\sqrt{2}}{2+\sqrt{2}} + \dfrac{2\sqrt{2}}{(2+\sqrt{2})^2}(z - \pi/4) - \dfrac{2\sqrt{2}+6}{(2+\sqrt{2})^3}$ $(z-\pi/4)^2 + \cdots$

37. $\dfrac{1}{2}\left(\dfrac{1}{e} - e\right) - \dfrac{3}{2}\left(\dfrac{1}{e} + e\right)(z-1) + \dfrac{9}{4}\left(\dfrac{1}{e} - e\right)$ $(z-1)^2 - \cdots$

39. $\dfrac{i}{4} + \dfrac{7}{16}(z-i) - \dfrac{25}{64}i(z-i)^2 - \dfrac{103}{256}(z-i)^3 + \cdots$

41. $1 - \dfrac{1}{4}x^2 - \dfrac{1}{96}x^4 - \dfrac{19}{5760}x^6 - \cdots$

43. $\dfrac{(1+i)}{\sqrt{2}}\left[1 - \dfrac{i}{2}z + \sum_{n=2}^{\infty}(-1)^{n-1}\dfrac{1\cdot 3\ldots(2n-3)}{2\cdot 4\ldots 2n}z^n\right]$

45. $4\pi i + z - \left(\dfrac{1}{2} + 2\pi i\right)z^2 - \dfrac{1}{6}z^3 + \cdots$

47. $\dfrac{1}{2}z^2 + z^3 + \dfrac{35}{24}z^4 + \cdots$

49. $1 + z - 2z^2 - 2z^3 + \cdots$

51. $z - \dfrac{z^3}{3} + \dfrac{z^5}{5} - \dfrac{z^7}{7} + \cdots$

53. $\displaystyle\int_0^z \frac{\sin u}{u}du = z - \frac{1}{18}z^3 + \frac{1}{600}z^5 - \cdots$ (divide the series for $\sin u$ by u and integrate the result term by term)

55. $z + z^2 + \dfrac{5}{6}z^3 + \dfrac{5}{6}z^4 + \cdots$

Exercise Set 15.3

1. $-\dfrac{1}{2}\displaystyle\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n$, $|z| < 2$

3. $\dfrac{1}{b-a}\displaystyle\sum_{n=0}^{\infty}\frac{b^{n+1} - a^{n+1}}{a^{n+1} + b^{n+1}}z^n, \quad |z| < |a|$

5. $\dfrac{1}{a-b}\displaystyle\sum_{n=0}^{\infty}\left(\frac{z^n}{b^{n+1}} + \frac{a^n}{z^{n+1}}\right), \quad |a| < |z| < |b|$

7. For $z = 0$;

$$f(z) = \exp[1/(1-z)] = \exp[-1/(z-1)]$$
$$= 1 - \frac{1}{(z-1)} + \frac{1}{2!(z-1)^2} - \frac{1}{3!(z-1)^3} + \cdots = \sum_{n=0}^{\infty}(-1)^n\frac{1}{n!(z-1)^n},$$
$$0 < |z-1| < \infty.$$

For $|z| > 1$; $f(z) = \exp[-1/(z-1)] = \exp[-\frac{1}{z}(1-\frac{1}{z})^{-1}]$. Now expand $(1-\frac{1}{z})^{-1}$ by the binomial theorem and multiply the result by $-1/z$ to obtain

$$f(z) = \exp\left[-\frac{1}{z} - \frac{1}{z^2} - \cdots\right]$$
$$= 1 - \left(\frac{1}{z} + \frac{1}{z^2} + \cdots\right) + \frac{1}{2!}\left(\frac{1}{z} + \frac{1}{z^2} + \cdots\right)^2$$
$$- \cdots = 1 - \frac{1}{z} - \frac{1}{z^2} + \cdots.$$

9. $$\sin\left(\frac{z}{1-z}\right) = -\sin\left(1 + \frac{1}{z-1}\right)$$
$$= -\sin 1\cos\left(\frac{1}{z-1}\right) - \cos 1\sin\left(\frac{1}{z-1}\right)$$

Now substitute $1/(z-1)$ into the series for sine and cosine to obtain

$$\sin\left(\frac{z}{1-z}\right)$$
$$= -\sin 1\left(1 - \frac{1}{2!(z-1)^2} + \frac{1}{4!(z-1)^4} - \cdots\right)$$
$$- \cos 1\left(\frac{1}{z-1} - \frac{1}{3!(z-1)^3} + \cdots\right)$$
$$= -\sum_{n=0}^{\infty}\frac{\sin\left(1 + \frac{1}{2}n\pi\right)}{n!(z-1)^n}, 0 < |z-1| < \infty.$$

11. $\dfrac{1}{3}\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}2^{n-1} - 1}{z^n}, \quad |z| > 2$

13. Expand $\sinh(1+u)$ as a Maclaurin series and then set $u = 1/z$ to obtain

$$\frac{1}{2}\left(e - \frac{1}{e}\right) + \frac{1}{2}\left(e + \frac{1}{e}\right)\frac{1}{z} + \frac{1}{4}\left(e - \frac{1}{e}\right)\frac{1}{z^2} - \cdots,$$
$$|z| > 0$$

15. Multiply the series for $\sin z$ and $\sin z/3$ and divide the result by z^3 to obtain

$$\frac{1}{3}\frac{1}{z} - \frac{5}{81}z + \frac{14}{3645}z^3 + \cdots, \quad |z| > 0$$

17. Simple poles at $z = 0$ and $z = \pm 2$

19. $z = 0$ is an essential singularity

21. Removable singularity at $z = 0$ obtained by defining $f(0) = 1$

23. $z = 1$ is an essential singularity

25. $z = (1 \pm 2k)\pi/2$, $k = 0, 1, \ldots$ are second order poles

27. Removable singularity at $z = 0$ obtained by defining $f(0) = -2$

29. $a_n = \frac{1}{2\pi i}\int_\Gamma \frac{f(\varsigma)}{(\varsigma - z)^{n+1}}d\varsigma$, $n = 0, \pm 1, \pm 2, \ldots$ where Γ is the circle $|z - z_0| = R$ with

$$R_1 < R < R_2. \text{ So } |a_n| \leq \frac{1}{2\pi}\int_0^{2\pi} \frac{|f(\varsigma)|}{|\varsigma - z^{n+1}|} R d\theta \leq \frac{1}{2\pi}\frac{M}{R^n}\int_0^{2\pi} d\theta = \frac{M}{R^n}.$$

31. $\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{a}{z}\right)^n$, $|z| > 3$ **33.** $-\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\frac{1}{z^{2n}}$, $|z| > 1$

35. $z = \infty$ is a regular point

37. $z = \infty$ is a limit point of poles

39. There is an essential singularity at $z = \infty$

Exercise Set 15.4

1. $\operatorname{Res}[z = 2] = 5/4$; $\operatorname{Res}[z = -2] = -1/4$

3. $\operatorname{Res}[z = 0] = 3$; $\operatorname{Res}[z = -1] = -2$

5. $\operatorname{Res}[z = 0] = -1$; $\operatorname{Res}[z = -1] = 0$

7. $\operatorname{Res}[z = n\pi] = (-1)^n(n^2\pi + 3)$, $n = 0, \pm 1, \pm 2, \ldots$

9. $\operatorname{Res}[z = (2n+1)\pi/2] = -1$, $n = 0, \pm 1, \pm 2, \ldots$

11. $\operatorname{Res}[z = (2n+1)\pi i] = -1$, $n = 0, \pm 1, \pm 2, \ldots$

13. $z = 0$ is a removable singularity so $\operatorname{Res}[z = 0] = 0$;
$\operatorname{Res}[z = n\pi i] = (-1)^n i \sinh n\pi$, $n = \pm 1, \pm 2, \ldots$

15. $\operatorname{Res}[z = 2] = 0$

17. $-\pi i/3$ **19.** $12\pi i$ **21.** $-\pi i/\sqrt{2}$

23. $-2\pi i/9$ **25.** $\pi(1 - e^{-2})$

27. $-2\pi i\{\cos 1 + i \sin 1\}$

29. $2\pi/(a^2 - 1)^{1/2}$ **31.** $\pi/\sqrt{2}$ **33.** $2\pi/(1 - a^2)$

Exercise Set 15.5

1. $\pi/(4a)$

3. $\pi/(2\sqrt{2})$

5. $\pi/18$

7. $\dfrac{\pi(1+a)}{4a^3e^a}$

9. $\dfrac{\pi}{(a^2 - b^2)}\left(\dfrac{e^{-b}}{b} - \dfrac{e^{-a}}{a}\right)$

11. $\dfrac{\pi}{2}\exp[-ma/\sqrt{2}]\cos(ma/\sqrt{2})$

13. π

15. $\dfrac{\pi}{2}(b - a)$

17. $\dfrac{\pi}{2b^2}(1 - e^{-ab})$

19. $3\pi/8$

21. $\dfrac{\pi}{4}[e^{-a} + \sin a]$

23. $\pi/\sqrt{2}$

25. $\pi/\sqrt{3}$

27. $\pi/3$

Exercise Set 16.1

1. $f(t) = (1/a^2)(1 - \cos at)$

3. $f(t) = (1/2)(t\cos t + t\sin t - \sin t)$

5. $f(t) = t^2/2 - t + 1 - e^{-t}$

7. $f(t) = \dfrac{1}{2a^2}\left(\dfrac{\sin at}{a} - t\cos at\right)$

9. $f(t) = \dfrac{\sqrt{3}}{2}\dfrac{\Gamma(2/3)}{\pi t^{2/3}}$

11. $f(t) = H(t-2)[\cosh(t-2) + \sinh(t-2)]$

13. $f(t) = \dfrac{1}{\sqrt{a}}\operatorname{erf}(\sqrt{at})$

15. $f(t) = \frac{e^{-at}}{\sqrt{b-a}}\operatorname{erf}(\sqrt{(b-a)t})$. Set $\mathcal{L}^{-1}\{1/\sqrt{s+b}\} = e^{-bt}/\sqrt{\pi t}$ and $\mathcal{L}^{-1}\{1/(s+a)\} = e^{-at}$ and use the convolution theorem followed by a change of variable)

Exercise Set 17.1

1. A $\pi/2$ counterclockwise rotation, a uniform magnification by a factor 2, and a shift of origin causing the point $z = 1 + i$ to map to the point $w = 1 + 2i$

3. $w = (1 - i)(1 + 2z)$

5. $w = (3 - 2i)z + 2i - 10$

7. As the transformation is linear it preserves shape, so a mapping of one strip onto the other is obtained by mapping a point on one side of the strip in the z-plane onto a point on one side of the strip in the w-plane, and then repeating the process by mapping a point on the other side of the strip in the z-plane onto a point on the other side of the strip in the w-plane. Only the correspondence between one pair of points is specified, namely the point $z = ik$ in the z-plane maps to the point $w = 0$ in the w-plane, so the transformation will not be unique. If we choose to map the point $z = i(k+h)$ on the top of the strip in the z-plane to the point $w = 1$ on the other side of the strip in the w-plane, we must solve the equations $0 = iak + b$ and $1 = ia(k+h) + b$, leading to the transformation $w = -(iz + k)/h$. A different choice of points will lead to a different

transformation between the two strips that still preserves the condition $w(ik) = 0$.

9. Family of circles $c(u^2 + v^2) + u + v = 0$ tangent to the straight line $v = -u$ at the origin

11. $w = i(1 + z)/(1 - z)$; interior of circle maps to upper-half of the w-plane

13. $w = (2i - z)/(2z + i)$; interior of circle maps to the interior of a circle

15. $x = c$ maps to circle $u^2 + v^2 = \exp(2\pi c/a)$; $y = k$ maps to radial line $v = u \tan \pi k/a$

17. $x = c$ maps to hyperbola $\frac{u^2}{\cos^2 \pi c/a} - \frac{v^2}{\sin^2 \pi c/a} = 1$; $y = k$ maps to ellipse $\frac{u^2}{\cosh^2 \pi k/a} + \frac{v^2}{\sinh^2 \pi k/a} = 1$

19. Write transformation as $w = \left(\frac{(1+z)(1-\bar{z})}{(1-z)(1-\bar{z}}\right)^2$ and use the fact that on the circle $z\bar{z} = |\bar{z}|^2 = |z|^2 = 1$. Then find how the semicircular boundary and the strip CA map and, finally, show that a point inside the semicircle maps to a point in the upper half of the w-plane.

Exercise Set 17.2

3. $$\phi(x, y) = \phi_4 + \frac{1}{\pi}\left[(\phi_1 - \phi_2)\text{Arctan}\left(\frac{y}{x - x_1}\right) + (\phi_2 - \phi_3)\text{Arctan}\left(\frac{y}{x - x_2}\right) + (\phi_3 - \phi_4)\text{Arctan}\left(\frac{y}{x - x_3}\right)\right]$$

5. $T(x, y) = 30 + \dfrac{240}{\pi}\text{Arctan}\left(\dfrac{2y}{1 - x^2 - y^2}\right)$

7. $\phi(x, y) = 320 - \dfrac{220}{\pi}\text{Arctan}\left(\dfrac{1 - x^2 - y^2}{2y}\right)$

9. $U\left(x^2 y - y^3 - \dfrac{3x^2 y - y^3}{(x^3 - 3xy^2)^2 + (3x^2 y - y^3)^2}\right)$ $= \text{constant}$

11. The equation of the streamline is $y(1 - \frac{1}{x^2+y^2}) =$ constant. As this equation is an even function of x, the streamlines are symmetric about the y-axis and $y' = 0$ for $x = 0$, $y \geq 1$. Far from the origin the streamlines are parallel to the x-axis. A bounding streamline lies along the x-axis and around the unit semicircle. Routine calculations show $y' > 0$ for $x < 0$ and $y' < 0$ for $x > 0$. Any streamline can be replaced by a boundary, so as the flow is steady any streamline $\psi =$ constant can represent a free surface.

13. The equipotentials $u = c$ in the w-plane are the hyperbolas $\frac{x^2}{\sin^2 c} - \frac{y^2}{\cos^2 c} = 1$ and the flux lines $v = k$ are the ellipses $\frac{x^2}{\cosh^2 k} + \frac{y^2}{\sinh^2 k} = 1$. In steady state heat conditions this represents a semi-infinite metal lamina with edge $A_\infty B$ at $T = 200$, edge CD_∞ at $T = 100$, with the edge BC insulated. The equipotentials become isotherms and flux lines become heat flow lines.

15. $T(x, y) = 450 - \dfrac{350}{\pi}\text{Arctan}\left(\dfrac{1 - x^2 - y^2}{2x}\right)$

Exercise Set 18.1

1. (a) Quasilinear first order
(b) Linear first order
(c) Nonlinear first order
(d) Semilinear first order
(e) Linear first order
(f) Nonlinear first order
(g) Linear second order
(h) Nonlinear second order

3. $u(x, y) = 4\exp[x - (x^2 - 2y)^{1/2}] - 2$, $x^2 \geq 2y$

5. $u(x, y) = \exp[x - (x^2 - 2y + 2)^{1/2}] - 2$, $x^2 \geq 2y - 2$

Exercise Set 18.2

1. $u(x, y) = x + \frac{1}{2}y$; global

3. $u(x, y) = x - y + 3$; global

5. $u(x, y) = \frac{1}{2}\sin x - \sin(2y - x)$; global

7. $u(x, y) = 1 + 2y - 4x - y^2 + 4xy - 3x^2$; global

9. $u(x, y) = 3x + \tan x^2 + \tan(\frac{1}{2}y - x^2)$ for (x, y) such that $\tan x^2$ and $\tan(\frac{1}{2}y - x^2)$ are both finite

11. $u(x, y) = (y - x)/(x^2 - xy + 1)$ for (x, y) such that $x^2 - xy + 1 \neq 0$

13. The solution in parametric form is $u = e^{-x} \sin\xi$, $y = \xi + (1 - e^{-x})\sin y$. An attempt to eliminate the parameter ξ leads to an implicit solution, so it is best to use the parametric form.

15. The parametric form of the solution is $u = 4\xi e^{-3x}$, $y = \xi + \frac{8}{3}\xi(1 - e^{-3x})$. In this case the parameter ξ can be eliminated to give the simple explicit solution $u(x, y) = 12y/(11e^{3x} - 8)$, for x such that the denominator does not vanish.

17. The solution in parametric form is $u = (3 + 2\xi)e^{-x}$, $y = \xi + (3 + 2\xi)(1 - e^{-x})$. In this case the

parameter ξ can be eliminated to give the simple explicit solution $u(x, y) = (2y + 3)/(3e^x - 2)$, for x such that the denominator does not vanish.

Exercise Set 18.3

1. $u(x, t) = e^{3t/2}\sin(2x - 4t)$

3. $u(x, t) = \frac{1}{2}e^{2t}\{\cos(x + 3t) + 1\} - \frac{1}{2}$

5. $u(x, t) = e^{x+4t} + 6t^2 + 3xt$

7. $u(x, t) = x(2e^t - 1)$

9. $u(x, t) = x(4e^t - 1)$

11. $u(x, t) = \frac{1}{3}x(4e^t - 1)$

13. $u(x, t) = \dfrac{\cos(x - t)}{1 - 2t\cos(x - t)}$; provided the denominator does not vanish

15. $u(x, t) = \dfrac{-2xe^t}{5 - 4e^t}$; for $0 \le t < \ln\frac{5}{4}$

17. $u(x, t) = \dfrac{4(1 + x)e^{-4t}}{1 + 3e^{-4t}}$

19. $u(x, t) = \dfrac{(3x - 1)(1 + t)}{1 + 3t + 3t^2 + t^3}$; for $t > -1$

Exercise Set 18.4

1. Write the equation in the conservation form $\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^{n+1}}{n+1}\right) = 0$. The shock condition is $\Lambda(t)[[u]] = \frac{1}{n+1}[[u^{n+1}]]$.

3. Riemann problem (b) has a shock solution because of the intersection of its characteristics. The conservation form of the equation is $\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(\frac{1}{3}u^3) = 0$, so the shock condition is $\Lambda(t)[[u]] = \frac{1}{3}[[u^3]]$, and hence the shock speed is seen to be given by $\Lambda(t) = \frac{(27-1)}{3(3-1)} = \frac{13}{3}$.

5. A similar problem was solved in Section 18.4 with the initial condition $u(x, 0) = \left\{\begin{smallmatrix} 0, & x<0 \\ 1, & x>0 \end{smallmatrix}\right.$. The solution of Exercise 5 follows from the solution given in Section 18.4 by replacing x by $x - 2$ to obtain $u = (x - 2)/t$. The solution lies in the region $t > 0$ bounded by the characteristic $x = 2$ and the characteristic $x = t + 2$.

Exercise Set 18.6

1. Elliptic **3.** Elliptic **5.** Parabolic

7. Elliptic; $\xi = \frac{1}{2}(x + y), \eta = x : u_{\xi\xi} + u_{\eta\eta} + \frac{3}{2}u_\xi + 3u_\eta + 1 = 0$

9. Hyperbolic; $\xi = 9x + y, \eta = x + y : u_{\xi\eta} = -\frac{1}{64}(9u_\xi + u_\eta)$

11. Parabolic; $\xi = y - 3x, \eta = x : u_{\eta\eta} = u - 5$

13. $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{2}{5} & \frac{4}{5} \\ 0 & \frac{4}{5} & \frac{8}{5} \end{bmatrix}$, $\lambda_1 = 2, \lambda_2 = 1, \lambda_3 = 0$, so the PDE is parabolic

$$\mathbf{Q} = \begin{bmatrix} 0 & 1/\sqrt{5} & 2/\sqrt{5} \\ 1 & 0 & 0 \\ 0 & 2/\sqrt{5} & -1/\sqrt{5} \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

so as $\xi = \mathbf{Q}\mathbf{x}$,
$\xi_1 = (1/\sqrt{5})x_2 + (2/\sqrt{5})x_3, \xi_2 = x_2, \xi_3 = (2/\sqrt{5})x_2 - (1/\sqrt{5})x_3$.
The PDE becomes $\frac{\partial^2 u}{\partial \xi_1^2} + \frac{1}{\sqrt{5}}\frac{\partial u}{\partial \xi_1} + \frac{2}{\sqrt{5}}\frac{\partial u}{\partial \xi_3} + 2u + \frac{1}{2} = 0$.

15. $\mathbf{A} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$, $\lambda_1 = 3, \lambda_2 = 3, \lambda_3 = 1$, so the PDE is elliptic

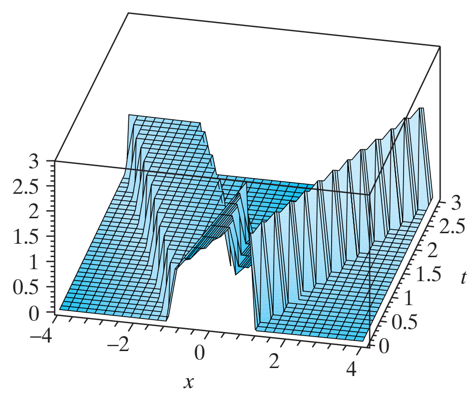
$$\mathbf{Q} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

so as $\xi = \mathbf{Q}\mathbf{x}$,
$\xi_1 = x_1, \ \xi_2 = -(1/\sqrt{2})x_2 + (1/\sqrt{2})x_3, \ \xi_3 = (1/\sqrt{2})x_2 + (1/\sqrt{2})x_3$. The PDE becomes $3u_{\xi_1\xi_1} + 3u_{\xi_2\xi_2} + u_{\xi_3\xi_3} + 4u - 7 = 0$. The further scaling $\zeta_1 = (1/\sqrt{3})\xi_1, \zeta_2 = (1/\sqrt{3})\xi_2, \zeta_3 = \xi_3$ reduces the PDE to the still simpler form $u_{\zeta_1\zeta_1} + u_{\zeta_2\zeta_2} + u_{\zeta_3\zeta_3} + 4u - 7 = 0$.

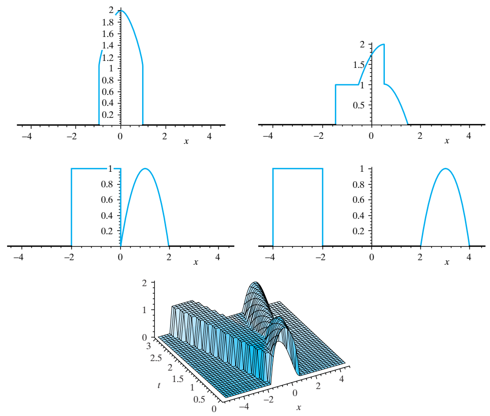
Exercise Set 18.8

In each case the solutions are given in the form of computer generated plots at the respective times $t = 0, t = 0.5, t = 1$ and $t = 3$. The 3D plot shown at the end of each solution illustrates how the waves evolve away from the initial condition.

1.



3.



Exercise Set 18.9

1. $f_{xx} = f''(x - ct)$, $g_{xx} = g''(x - ct)$, $f_{tt} = c^2 f''(x - ct)$, $g_{tt} = c^2 g''(x - ct)$, so $u = f + g$ satisfies $u_{tt} = c^2 u_{xx}$

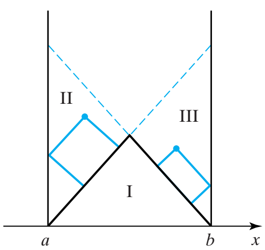
3. $u(x,t) = \dfrac{1}{2}\{\sin(x - ct) + \sin(x + ct)\} + \dfrac{1}{2c}\displaystyle\int_{x-ct}^{x+ct} \frac{ds}{1+s^2}$, and so $u(x,t) = \sin x \cos ct + \dfrac{1}{2c}\{\text{Arctan}(x + ct) - \text{Arctan}(x - ct)\}$

4. $u(x,t) = 1 + \dfrac{1}{2c}\displaystyle\int_{x-ct}^{x+ct} \cos s\, ds = 1 + \dfrac{1}{c}\cos c \sin ct$

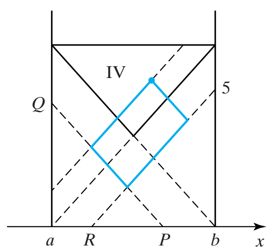
5. $u(x,t) = \dfrac{1}{2}\{\tanh(x - ct) + \tanh(x + ct)\} + \dfrac{1}{2c}\{\tanh(x + ct) - \tanh(x - ct)\}$, and so $u(x,t) = \left(\dfrac{c+1}{2c}\right)\tanh(x + ct) + \left(\dfrac{c-1}{2c}\right)\tanh(x - ct)$

6. $u(x,t) = \dfrac{1}{2}\{e^{x-ct} + e^{x+ct}\} + \dfrac{1}{2c}\displaystyle\int_{x-ct}^{x+ct} e^{-s} ds = e^x \cosh ct + \dfrac{1}{c}\sinh ct$

7.



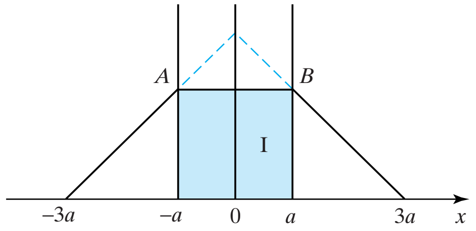
Use D'Alembert in (I), then (128) to find u in (II) and (III)



Use solutions in (I), (II) and (III) and (128) with characteristics PQ and RS to find solution in (IV)

The situation is now back to the original problem and so can be continued as long as necessary. This is a *theoretical* rather than a practical way of solving the problem.

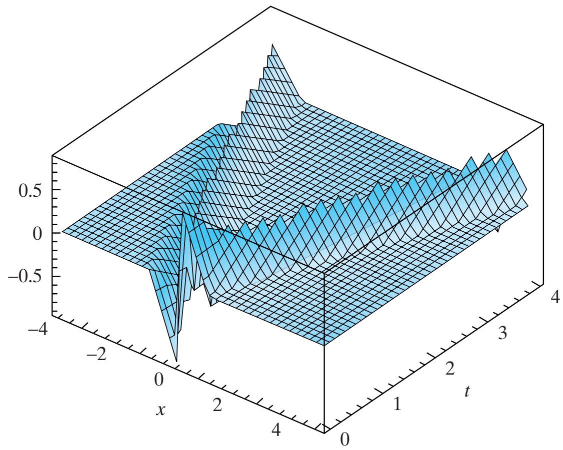
9.



Reflect the initial conditions as odd functions about $x = -a$ and $x = a$. Then the initial conditions are known for $-3a \le x \le 3a$. D'Alembert's formula can now be used to find the solution in (I). The solution is then known along AB, so the argument can be repeated using the conditions along AB as new initial conditions, etc.

11. From D'Alembert's formula with $g(x) \equiv 0$ we have

$$u(x,t) = \frac{1}{2}\{f(x - ct) + f(x + ct)\}.$$



13. $u(x, 1/4) = \dfrac{1}{2}\displaystyle\int_{x-1/4}^{x+1/4} g(s)ds$, so

$$u\left(x, \frac{1}{4}\right) = \begin{cases} 0, & x < -5/4 \\ \frac{1}{2}\int_{-1}^{x+1/4}(1 - s^2)ds, & -5/4 \le x \le -3/4 \\ \frac{1}{2}\int_{x-1/4}^{x+1/4}(1 - s^2)ds, & -3/4 \le x \le 3/4 \\ \frac{1}{2}\int_{x-1/4}^{1}(1 - s^2)ds, & 3/4 \le x \le 5/4 \\ 0, & x > 5/4 \end{cases}$$

Exercise Set 18.10

1. $u(x,t)=\dfrac{4kL^2}{\pi^3}\sum_{n=1}^{\infty}\dfrac{1}{n^3}[2(-1)^{n+1}-1]$
$\times\ \sin\dfrac{n\pi x}{L}\cos\dfrac{n\pi ct}{L}$

3. $u(x,t)=\dfrac{8k}{\pi^2}\sum_{n=1}^{\infty}\dfrac{1}{n^2}\sin\dfrac{n\pi}{2}\sin\dfrac{n\pi x}{L}\cos\dfrac{n\pi ct}{L}$

5. $u(x,t)=k\sin\dfrac{\pi x}{L}\cos\dfrac{2\pi ct}{L}$

7. $u(x,t)=\dfrac{4kL^3}{\pi^2}\sum_{n=1}^{\infty}\dfrac{1}{n^4}[1+(-1)^{n+1}]$
$\times\ \sin\dfrac{n\pi x}{L}\sin\dfrac{n\pi ct}{L}$

9. $u(x,t)=\dfrac{8k}{\pi^2}\sum_{n=0}^{\infty}\dfrac{(-1)^n}{(2n+1)^2}\sin\dfrac{(2n+1)\pi x}{2L}$
$\times\ \cos\dfrac{(2n+1)\pi ct}{2L}$

11. $u(x,y)=\dfrac{e^{-6x}}{13}(4e^{-9y}+9e^{4y})$. When $y\gg 0$,
$u(x,y)\approx\dfrac{9}{13}\exp(4y-6x)$.

13. $u(x,t)=\dfrac{L}{2}-\dfrac{4L}{\pi^2}\sum_{n=1}^{\infty}\exp\left[-\dfrac{(2n-1)^2\pi^2kt}{L^2}\right]$
$\times\ \dfrac{1}{(2n-1)^2}\cos\dfrac{(2n-1)\pi x}{L}$

17. $u(x,y,t)=2\sin\frac{3\pi x}{c}\sin\left(\frac{\pi y}{d}\right)\cos(\pi t\sqrt{(3/c)^2+(1/d)^2})$. The initial condition is an eigenfunction.

21. $T(x,y)=1-\dfrac{1}{2}e^{-x}\cos y+2\sum_{n=2}^{\infty}(-1)^n$
$\times\ \dfrac{(1-n^2)}{(1-2n^2+n^4)}\exp(-nx)\cos(ny)$

Exercise Set 18.12

1. Taking the Laplace transform of the PDE with respect to t gives $s\bar{T}-T_0=\kappa\frac{d^2\bar{T}}{dx^2}$ and $\bar{T}(0,s)=0$ with the general solution $\bar{T}(x,s)=A\exp(\sqrt{\frac{s}{\kappa}}x)+B\exp(-\sqrt{\frac{s}{\kappa}}x)+\frac{T_0}{s}$. The solution can only be finite for all x if $A=0$, so $\bar{T}(x,s)=T_0\left\{\frac{1-\exp\left(-x\sqrt{s/\kappa}\right)}{s}\right\}$. Finding the inverse of this transform then gives $T(x,t)=T_0\mathrm{erf}\left[\frac{x}{2\sqrt{\kappa t}}\right]$.

3. Taking the Laplace transform of the PDE with respect to t gives $s\bar{T}-T_0=\kappa\frac{d^2\bar{T}}{dx^2}$ so the general solution is $\bar{T}(x,s)=A\exp(x\sqrt{s/\kappa})+B\exp(-x\sqrt{s/\kappa})+T_0/s$. The solution can only be bounded for all x if $A=0$, so $\bar{T}(x,s)=B\exp(-x\sqrt{s/\kappa})+T_0/s$. Now $\mathcal{L}\{T(0,t)\}=T_0s/(s^2+a^2)$ so setting $x=0$ in the above result gives $T_0s/(s^2+a^2)=B+T_0/s$ so that

$$\bar{T}(x,s)=T_0\left(\frac{s}{s^2+a^2}\right)\exp\left(-x\sqrt{\frac{s}{\kappa}}\right)-T_0\frac{1}{s}\exp\left(-x\sqrt{\frac{s}{\kappa}}\right)+\frac{T_0}{s}.$$

Using the convolution theorem to invert the transform gives

$$T(x,t)=\frac{T_0}{2\sqrt{\pi\kappa}}\int_0^t\frac{\cos a\tau}{(t-\tau)^{3/2}}\exp\left(\frac{-x^2}{4\kappa(t-\tau)}\right)d\tau-\frac{T_0x}{2\sqrt{\pi\kappa}}\int_0^t\frac{1}{\tau^3}\exp\left(\frac{-x^2}{4\kappa\tau}\right)d\tau+T_0.$$

5. Taking the Fourier transform of the PDE with respect to x gives $\bar{u}_{tt}(\omega,t)+(k+c^2\omega^2)\bar{u}(\omega,t)=0$, and so $\bar{u}(\omega,t)=a(\omega)\cos(t\sqrt{k+c^2\omega^2})+b(\omega)\times\sin(t\sqrt{k+c^2\omega^2})$, showing that $\bar{u}_t(\omega,t)=\sqrt{k+c^2\omega^2}\left\{-a(\omega)\sin(t\sqrt{k+c^2\omega^2})+b(\omega)\times\cos(t\sqrt{k+c^2\omega^2})\right\}$.

From the initial conditions

$$a(\omega)=\bar{u}(\omega,0)=\frac{U}{\sqrt{2\pi}}\int_{-1}^{1}e^{-i\omega x}dx=U\sqrt{\frac{2}{\pi}}\frac{\sin\omega}{\omega},\ \sqrt{k+c^2\omega^2}b(\omega)=\bar{u}_t(\omega,0)=0,\ \text{so that } \bar{u}(\omega,t)=U\sqrt{\frac{2}{\pi}}\frac{\sin\omega}{\omega}\cos(t\sqrt{k+c^2\omega^2}).$$

Taking the inverse transform then gives

$$u(x,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\bar{u}(\omega,t)e^{i\omega x}d\omega=\frac{2U}{\pi}\int_0^{\infty}\frac{\sin\omega}{\omega}\cos(t\sqrt{k+c^2\omega^2})\cos\omega x\,d\omega.$$

7. Take the Fourier transform with respect to x of the PDE to obtain $-\omega\bar{u}(\omega,y)+\frac{d^2\bar{u}}{dy^2}=0$ for $y>0$, where $\bar{u}(\omega,0)=F(\omega)$, the transform of $f(x)$. For the solution to remain bounded when y is large it then follows that $\bar{u}(\omega,y)=F(\omega)e^{-|\omega|}y$. Taking

the inverse transform then gives

$$u(x, y) = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{f(\tau)}{y^2 + (x-\tau)^2} d\tau.$$

9. Differentiate the result with respect to x and expand $e^{i\omega x}$ by de Moivre's theorem. The integral containing $\omega \cos \omega x$ vanishes because this is an odd function of ω and the remaining integral containing the function $\omega \sin \omega x$ is an even function of ω, so the result follows from the definition of the sine transform after changing the interval of integration to $[0, \infty)$.

11. Proceed as in the heat conduction example in Section 10.2 using the given form of $T(x, 0)$. The solution reduces to $T(x,t) = \frac{1}{2}T_0\{\mathrm{erf}[(x + a)/(2\sqrt{\kappa t})] - \mathrm{erf}[(x - a)/(2\sqrt{\kappa t})]\}$.

Exercise Set 19.2

1. 2.27886

3. 1.40619

5. −1.08601

7. $x_{r+1} = \frac{1}{2}(x_r + a/x_r^{n-1})$

9. −1.08090, 2.54109, 2.83981

11. 0.67567

13. 2.84387

15. 3.70665

17. 0.25763

Exercise Set 19.4

1. $I = 28$ **3.** $I_{\text{trap}} = 1.849317$, $I_{\text{simp}} = 1.851944$, $I_{\text{exact}} = 1.851937$

5. 0.596584

7. $J_1(2) = 0.576725$ (the result obtained by Simpson's rule agrees with the exact result to six decimal places)

9. $J_1(4) = -0.065743$ (using Simpson's rule)

11. $I_0(3.5) = 7.378203$ (the result obtained by Simpson's rule agrees with the exact result to six decimal places)

Exercise Set 19.5

1. $x_1 = 0.73826$, $x_2 = -0.73918$, $x_3 = 0.75556$ (Gaussian elimination)

3. $x_1 = -0.90034$, $x_2 = -1.14831$, $x_3 = -0.95315$ (Gaussian elimination)
Not diagonally dominant: interchange first and second equations

5. $x_1 = -66.51395$, $x_2 = 927.64721$, $x_3 = -2585.93671$, $x_4 = 1862.64259$

When calculations are rounded to five decimal places det $\mathbf{H}_4 = 1.6111 \times 10^{-7}$. Exact value det $\mathbf{H}_4 = 1/6048000 \approx 1.65344 \times 10^{-7}$

7. $\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix}$, $\mathbf{U} = \begin{bmatrix} -4 & 1 & -1 \\ 0 & 2 & 2 \\ 0 & 0 & -3 \end{bmatrix}$,
$x_1 = -53/24$, $x_2 = -7/6$, $x_3 = 14/3$

9. $\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ -1 & 3 & 1 \end{bmatrix}$, $\mathbf{U} = \begin{bmatrix} 4 & -1 & -1 \\ 0 & 2 & -3 \\ 0 & 0 & -1 \end{bmatrix}$,
$x_1 = 131/8$, $x_2 = 81/2$, $x_3 = 25$

11. $\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/2 & 1 & 0 & 0 \\ 2 & -1 & 1 & 0 \\ -1 & 2 & 2 & 1 \end{bmatrix}$, $\mathbf{U} = \begin{bmatrix} 2 & 1 & 0 & 2 \\ 0 & 1/2 & 1 & 1 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
$x_1 = -3/2$, $x_2 = 10$, $x_3 = 1/2$, $x_4 = -3$

Exercise Set 19.6

1. $\lambda = 19.24435$ (exact), $\tilde{\mathbf{x}} = [1, 0.41089, -0.01169]^{\mathrm{T}}$

3. $\lambda = 28.19020$ (exact), $\tilde{\mathbf{x}} = [0.07079, 0.04865, 1]^{\mathrm{T}}$

5. $\lambda = 27.35196$ (exact), $\tilde{\mathbf{x}} = [1, 0.42720, 0.07037]^{\mathrm{T}}$

7. $\lambda = 2.55051$ (exact), $\mathbf{x} = [1.44949, -1, 1]^{\mathrm{T}}$ (not normalized)

9. $\lambda = -3.04390$ (exact), $\mathbf{x} = [-4.68367, 1, 4.94464]^{\mathrm{T}}$ (not normalized)

Exercise Set 19.7

1.

x_n	2.0	2.2	2.4	2.6	2.8	3.0
y_n	0	0.66419	1.28937	1.89393	2.48875	3.08063

3.

x_n	1.0	1.2	1.4	1.6	1.8	2.0
y_n	2.0	2.17043	2.27255	2.29924	2.25314	2.14619

5.

x_n	1.0	1.2	1.4	1.6	1.8	2.0
y_n	1.0	0.40577	0.08015	−0.10414	−0.20593	−0.24801

7.

x_n	0	0.1	0.2	0.3	0.4	0.5
y_n	2.0	1.87998	1.71971	1.51888	1.27772	0.99787

9.

x_n	0	0.1	0.2	0.3	0.4	0.5
y_n	1.0	1.07995	1.12053	1.12465	1.09709	1.04377

11.

x_n	0	0.2	0.4	0.6	0.8	1.0
y_n	2.0	2.24068	2.57043	3.01382	3.61800	4.46785

13.

x_n	1.0	1.2	1.4	1.6	1.8	2.0
y_n	1.0	1.23999	1.55909	1.95332	2.41386	2.92755

15.

x_n	0	0.2	0.4	0.6	0.8	1.0
y_n	−2.0	−1.72167	−1.25453	−0.72717	−0.01088	0.90446

17.

t_n	0	0.2	0.4	0.6	0.8	1.0
x_n	1.0	1.00348	1.02480	1.07075	1.17222	1.32949
y_n	0	−0.18397	−0.35117	−0.52343	−0.72280	−0.97510

19.

t_n	0	0.2	0.4	0.6	0.8	1.0
x_n	1.0	0.80588	0.64974	0.55084	0.49643	0.46921
y_n	1.0	0.87511	0.79475	0.74186	0.70938	0.69102

REFERENCES

General References

[G.1] M. Abrabowitz and I. A. Stegun (Eds.), *Handbook of Mathematical Functions*, Dover (reprint), New York, 1970

[G.2] I. S. Gradshteyn and I. M. Ryzhik (Ed. A. Jeffrey), *Tables of Integrals, Series, and Products*, 6th ed., Academic Press, Boston, 2000

[G.3] A. Jeffrey, *Handbook of Mathematical Formulas and Integrals*, Academic Press, New York, 1995

Part One Review Material

[1.1] I. D. Faires and B. T. Faires, *Calculus*, 2nd ed., Random House, New York, 1988

[1.2] R. L. Finney and G. B. Thomas Jr., *Calculus and Analytic Geometry*, 9th ed., Addison-Wesley, Reading, MA, 1996

[1.3] R. E. Larson, R. P. Hostetler, and B. E. Edwards, *Calculus with Analytic Geometry*, 4th ed., D. C. Heath, Lexington, MA, 1990

[1.4] J. E. Marsden and A. J. Tromba, *Vector Calculus*, 2nd ed., W. H. Freeman, San Francisco, 1981

[1.5] M. H. Protter and P. E. Protter, *Calculus with Analytic Geometry*, 4th ed., Jones and Bartlett, Boston, 1988

[1.6] M. H. Protter and C. B. Morrey, Jr., *Modern Mathematical Analysis*, Addison-Wesley, Reading, MA, 1964

[1.7] D. G. Zill, *Calculus with Analytic Geometry*, 2nd ed., PWS-Kent, Boston, 1988

Part Two Vectors and Matrices

[2.1] H. Anton and C. Rorres, *Elementary Linear Algebra: Applications Version*, 6th ed., Wiley, New York, 1991

[2.2] K. P. Bogart, *Introductory Combinatorics*, Pitman, London, 1983

[2.3] D. E. Bourne and P. C. Kendall, *Vector Analysis and Cartesian Tensors*, 3rd ed., Chapman and Hall, London, 1992

[2.4] A. B. Clarke and R. L. Disney, *Probability and Random Processes for Engineers and Scientists,* Wiley, New York, 1970

[2.5] C. G. Cullen, *Linear Algebra with Applications*, 2nd ed., Addison-Wesley, Reading, MA, 1997

[2.6] R. L. Finney and G. B. Thomas Jr., *Calculus*, 2nd ed., Addison-Wesley, Reading, MA, 1994

[2.7] S. I. Grossman, *Elementary Linear Algebra*, 3rd ed., Wadsworth, Belmont, CA, 1987

[2.8] L. Mirsky, *An Introduction to Linear Algebra*, Oxford University Press, Oxford, 1963

[2.9] P. V. O'Neal, *Introduction to Linear Algebra: Theory and Applications*, Wadsworth, Belmont, CA, 1979

[2.10] E. D. Nering, *Linear Algebra and Matrix Theory*, Wiley, New York, 1970

[2.11] B. Nobel and J. W. Daniel, *Applied Linear Algebra*, 3rd ed., Prentice-Hall, Englewood Cliffs, NJ, 1988

[2.12] G. Strang, *Linear Algebra and its Applications*, 2nd ed., Academic Press, New York, 1980

[2.13] S. A. Wiitala, *Discrete Mathematics: A Unified Approach*, McGraw-Hill, New York, 1987

[2.14] K. E. Atkinson, *An Introduction to Numerical Analysis*, 2nd ed., Wiley, New York, 1989

[2.15] G. J. Borse, *Numerical Mathematics with MATLAB*, PWS, Boston, 1997

[2.16] W. Cheney and D. Kincaid, *Numerical Mathematics and Computing*, Brooks/Cole, San Francisco, 1994

[2.17] C. E. Froberg, *Numerical Mathematics*, Benjamin Cummings, Menlo Park, CA, 1985

[2.18] L. W. Johnson and R. D. Riess, *Numerical Analysis*, 2nd ed., Addison-Wesley, Reading, MA, 1982

[2.19] W. H. Press, B. P. Flannen, S. A. Teukolsky, and W. T. Vetterling, *Numerical Recipes: The Art of Scientific Computing*, Cambridge University Press, Cambridge, UK, 1987

[2.20] J. Stoer and R. Bulirsch, *Introduction to Numerical Analysis*, Springer-Verlag, New York, 1980

Part Three Ordinary Differential Equations

[3.1] V. I. Arnold, *Ordinary Differential Equations*, Springer-Verlag, New York, 1992

[3.2] D. K. Arrowsmith and C. M. Place, *Dynamical Systems and Nonlinear Ordinary Differential Equations*, Chapman and Hall, London, 1995

[3.3] G. Birkhoff and Gian-Carlo Rota, *Ordinary Differential Equations*, 4th ed., Wiley, New York, 1989

[3.4] W. E. Boyce and R. C. DiPrima, *Elementary Differential Equations and Boundary Value Problems*, 3rd ed., Wiley, New York, 1977

[3.5] F. Brauer and J. A. Nohel, *Introduction to Differential Equations with Applications*, Harper and Row, New York, 1986

[3.6] M. Braun, *Differential Equations and Their Applications*, Springer-Verlag, New York, 1975

[3.7] J. W. Brown and R. V. Churchill, *Fourier Series and Boundary Value Problems*, 5th ed., McGraw-Hill, 1993

[3.8] H. S. Carslaw and J. C. Jaeger, *Operational Methods in Applied Mathematics*, 2nd ed., Oxford University Press, London, 1949

[3.9] R. V. Churchill, *Operational Methods*, 3rd ed., McGraw-Hill, New York, 1972

[3.10] E. A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations*, McGraw-Hill, New York, 1955

[3.11] A. Erdelyi, W. Magnus, F. Oberhettinger and F. Tricomi, *Tables of Integral Transforms*, Vols. I and II, McGraw-Hill, New York, 1954

[3.12] E. L. Ince, *Ordinary Differential Equations*, Dover (reprint), New York, 1956

[3.13] D. W. Jordan and P. Smith, *Nonlinear Ordinary Differential Equations*, 3rd ed., Clarendon Press, Oxford, 1999

[3.14] F. Oberhettinger and L. Bandii, *Tables of Laplace Transforms*, Springer-Verlag, New York, 1973

[3.15] M. Krusemeyer, *Differential Equations*, Macmillan, New York, 1994

[3.16] S. L. Ross, *Differential Equations*, 3rd ed., Wiley, New York, 1984

[3.17] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, 2nd ed., Cambridge University Press, Cambridge, UK, 1966

[3.18] D. V. Widder, *The Laplace Transform*, Princeton University Press, Princeton, NJ, 1941

[3.19] D. G. Zill, *A First Course in Differential Equations with Applications*, 3rd ed., PWS, Boston, 1986

[3.20] K. E. Atkinson, *An Introduction to Numerical Analysis*, 2nd ed., Wiley, New York, 1989

[3.21] G. J. Borse, *Numerical Mathematics with MATLAB*, PWS, Boston, 1997

[3.22] W. Cheney and D. Kincaid, *Numerical Mathematics and Computing*, Brooks/Cole, San Francisco, 1994

[3.23] C. E. Froberg, *Numerical Mathematics*, Benjamin Cummings, Menlo Park, California, 1985

[3.24] L. W. Johnson and R. D. Riess, *Numerical Analysis*, 2nd ed., Addison-Wesley, Reading, MA, 1982

[3.25] J. L. Morris, *Computational Methods in Elementary Numerical Analysis*, Wiley, New York, 1983

[3.26] A. Ralston and P. Rabinowitz, *A First Course in Numerical Analysis*, 2nd ed., McGraw-Hill, New York, 1978

Part Four Fourier Series, Integrals, and the Fourier Transform

[4.1] W. Brown and R. V. Churchill, *Fourier Series and Boundary Value Problems*, 5th ed., McGraw-Hill, New York, 1993

[4.2] A. Erdelyi, W. Magnus, F. Oberhettinger, and F. Tricomi, *Tables of Integral Transforms*, Vols. I and II, McGraw-Hill, New York, 1954

[4.3] I. N. Sneddon, *Fourier Transforms*, McGraw-Hill, New York, 1951

[4.4] I. N. Sneddon, *The Use of Integral Transforms*, McGraw-Hill, New York, 1972

[4.5] A. Zygmund, *Trigonometric Series*, 2nd ed. (Volumes I and II combined), Cambridge University Press, Cambridge, UK, 1988

Part Five Vector Calculus

[5.1] P. Baxandall and H. Liebeck, *Vector Calculus*, Oxford University Press, Oxford, 1986

[5.2] D. E. Bourne and P. C. Kendall, *Vector Analysis and Cartesian Tensors*, 3rd ed., Chapman and Hall, London, 1992

[5.3] J. E. Marsden and A. J. Tromba, *Vector Calculus*, 2nd ed., W. H. Freeman, San Francisco, 1981

[5.4] G. E. Mase and G. T. Mase, *Continuum Mechanics for Engineers*, CRC Press, Boca Raton, FL, 1992

[5.5] M. H. Protter and C. B. Morrey, Jr., *Modern Mathematical Analysis*, Addison-Wesley, Reading, MA, 1964

[5.6] M. R. Spiegel, *Vector Analysis*, Schaum Outline Series, McGraw-Hill, New York, 1974

Part Six Complex Analysis

[6.1] R. V. Churchill and J. W. Brown, *Complex Variables and Applications*, 5th ed., McGraw-Hill, New York, 1990

[6.2] P. Henrici, *Applied Computational Complex Analysis* (3 volumes), Wiley, New York, 1977, 1988, 1991

[6.3] J. E. Marsden, *Basic Complex Analysis*, Freeman, San Francisco, 1973

[6.4] J. H. Mathews and R.W. Howell, *Complex Analysis for Mathematics and Engineering*, 3rd ed., Jones and Bartlett, Boston, 1997

[6.5] L. M. Milne-Thompson, *Theoretical Hydrodynamics*, 5th ed., Macmillan, London, 1972

[6.6] J. D. Paliouras and D. S. Meadows, *Complex Variables for Scientists and Engineers*, 2nd ed., Macmillan, New York, 1975

[6.7] L. Pennisi, *Elements of Complex Variables*, 2nd ed., Holt, Rinehart and Winston, New York, 1976

[6.8] L. R. Rubenfeld, *A First Course in Applied Complex Variables*, Wiley, New York, 1985

[6.9] E. B. Saff and A. D. Snider, *Fundamentals of Complex Analysis for Mathematicians, Scientists and Engineers*, 2nd ed., Prentice Hall, Englewood Cliffs, NJ, 1993

[6.10] J. L. Schiff, *The Laplace Transform: Theory and Applications*, Springer-Verlag, New York, 1999

Part Seven Partial Differential Equations

[7.1] D. R. Bland, *Wave Theory and Applications*, Clarendon Press, Oxford, 1988

[7.2] J. W. Brown and R. V. Churchill, *Fourier Series and Boundary Value Problems*, 5th ed., McGraw-Hill, New York, 1993

[7.3] C. A. Coulson and A. Jeffrey, *Waves: A Mathematical Approach to the Common Types of Wave Motion*, 2nd ed., Longman, London, 1977

[7.4] R. Courant and K. O. Friedrichs, *Supersonic Flow and Shock Waves*, Wiley-Interscience, New York, 1956

[7.5] G. F. D. Duff and D. Naylor, *Differential Equations of Applied Mathematics*, Wiley, New York, 1966

[7.6] P. R. Garabedian, *Partial Differential Equations*, Wiley, New York, 1964

[7.7] R. Haberman, *Elementary Applied Partial Differential Equations*, 2nd ed., Prentice Hall, Englewood Cliffs, NJ, 1983

[7.8] R. Knobel, *An Introduction to the Mathematical Theory of Waves*, Student Mathematical Library Volume 3, American Mathematical Society, Rhode Island, 1999

[7.9] R. J. LeVeque, *Numerical Methods for Conservation Laws*, Birkhauser, Boston, 1990

[7.10] H. Levine, *Partial Differential Equations, Studies in Advanced Mathematics*, Vol. 6, American Mathematical Society, Rhode Island, 1991

[7.11] J. D. Logan, *Applied Partial Differential Equations*, Springer-Verlag, Berlin, 1998

[7.12] P. V. O'Neal, *Beginning Partial Differential Equations*, Wiley, New York, 1999

[7.13] J. Smoller, *Shock Waves and Reaction–Diffusion Equations*, Springer-Verlag, Berlin, 1983

[7.14] I. N. Sneddon, *The Use of Integral Transforms*, McGraw-Hill, New York, 1972

[7.15] W. A. Strauss, *Partial Differential Equations: An Introduction*, Wiley, New York, 1992

[7.16] M. E. Taylor, *Partial Differential Equations: Basic Theory*, Springer-Verlag, New York, 1996

[7.17] J. L. Troutman, *Boundary Value Problems of Applied Mathematics*, PWS, Boston, 1994

[7.18] G. B. Whitham, *Linear and Nonlinear Waves*, Wiley, New York, 1974; reprinted by Wiley, New York, 1999

[7.19] E. C. Zachmanoglou and D. W. Thoe, *Introduction to Partial Differential Equations with Applications*, Williams and Wilkins, Baltimore, 1976

[7.20] E. Zauderer, *Partial Differential Equations of Applied Mathematics*, 2nd ed., Wiley, New York, 1989

Part Eight Numerical Mathematics

[8.1] K. E. Atkinson, *An Introduction to Numerical Analysis*, 2nd ed., Wiley, New York, 1989

[8.2] G. J. Borse, *Numerical Mathematics with MATLAB*, PWS, Boston, 1997

[8.3] W. Cheney and D. Kincaid, *Numerical Mathematics and Computing*, Brooks/Cole, San Francisco, 1994

[8.4] C. E. Fröberg, *Numerical Mathematics*, Benjamin Cummings, Menlo Park, CA, 1985

[8.5] L. W. Johnson and R. D. Riess, *Numerical Analysis*, 2nd ed., Addison-Wesley, Reading, MA, 1982

[8.6] R. J. LeVeque, *Numerical Methods for Conservation Laws*, Birkhauser, Boston, 1990

[8.7] J. Ll. Morris, *Computational Methods in Elementary Numerical Analysis*, Wiley, New York, 1983

[8.8] J. M. Ortega and W. G. Poole, Jr., *An Introduction to Numerical Methods for Differential Equations*, Pitman, London, 1981

[8.9] A. Ralston and P. Rabinowitz, *A First Course in Numerical Analysis*, 2nd ed., McGraw-Hill, New York, 1978

Suggested Further Reading

Linear Algebra

F. Chatelin, *Eigenvalues of Matrices*, Wiley-Interscience, New York, 1993

G. H. Golub and C. F. Van Loan, *Matrix Computations*, 3rd ed., Baltimore, Johns Hopkins University Press, 1996

J. H. Wilkinson, *The Algebraic Eigenvalue Problem*, Clarendon Press, Oxford, 1988

Analytic Functions

E. C. Titchmarsh, *The Theory of Functions*, 2nd ed., Oxford University Press, 1975 (reprint)

Applied Mathematics and Differential Equations

H. S. Carslaw and J. C. Jaeger, *Conduction of Heat in Solids*, 2nd ed., Clarendon Press, Oxford, 1986 (reprint)

J. P. Keener, *Principle of Applied Mathematics: Transformation and Approximation*, Addison-Wesley, Reading, MA, 1988

J. D. Logan, *Applied Mathematics: A Contemporary Approach*, Wiley-Interscience, New York, 1981

J. D. Logan, *An Introduction to Nonlinear Partial Differential Equations*, Wiley-Interscience, New York, 1994

D. Zwillinger, *Handbook of Differential Equations*, 2nd ed., Academic Press, Boston, 1992

Numerical Analysis

L. F. Shampine, *Numerical Solution of Ordinary Differential Equations*, Chapman and Hall, New York, 1994

L. F. Shampine et al., *Fundamentals of Numerical Computation*, Wiley-Interscience, New York, 1996

INDEX

D

E

G

H

I

J

K

L

M

N

O

P

Q

R

S

T